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Advances in perturbative thermal field theory

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Abstract. The progress of the last decade in perturbative quantum field theory at high temperature and density made possible by the use of effective field theories and hard-thermal/dense-loop resummations in ultrarelativistic gauge theories is reviewed. The relevant methods are discussed in field theoretical models from simple scalar theories to non-Abelian gauge theories including gravity. In the simpler models, the aim is to give a pedagogical account of some of the relevant problems and their resolution, while in the more complicated but also more interesting models such as quantum chromodynamics, a summary of the results obtained so far are given together with references to a few most recent developments and open problems.

Contents

1. Introduction

Ultrarelativistically hot and dense matter needs to be understood in many problems ranging from early-universe cosmology [1, 2] as well as the astrophysics of compact stars [3] to the current and prospected experimental programs at relativistic heavy-ion colliders in the USA (RHIC at Brookhaven) and in Europe (LHC at CERN), which seek to produce and investigate the new state of matter called the quark-gluon plasma [4]. The theoretical framework for analysing equilibrium and near-equilibrium properties is thermal field theory, which at a fundamental level involves Abelian and non-Abelian gauge theories.

In this review we shall focus on the technical difficulties to derive results by analytical, weak-coupling techniques. Because of asymptotic freedom of non-Abelian gauge theories, this is a valid approach at least at sufficiently high temperatures and densities. Much of the present activities is concerned with quantum chromodynamics (QCD), which is however strongly interacting at all temperatures of experimental interest. The perturbative calculations that have been carried out to high loop orders for thermodynamic potentials seem to signal that perturbative QCD at finite temperature is loosing any predictive power below ridiculously high temperatures $\sim 10^5$ GeV, where the coupling is so small that everything is described well enough by an ideal gas of quarks and gluons without much need of perturbative refinements. However, recent studies indicate that this conclusion is much too pessimistic, and it appears that even in strongly coupled QCD it is indeed possible (at least to some extent) to employ weak-coupling methods if proper use is made of effective field theories and the fact that the nontrivial spectral data of quasiparticles can absorb much of the strong interactions [5].

In the case of static quantities at high temperature and small chemical potential the relevant techniques are effective field theories which take advantage of the phenomenon of dimensional reduction; for dynamical quantities the required techniques involve the effective theory produced by hard thermal loops [6]. At high chemical potential and low temperature, the resummation of hard dense loops becomes important even in static quantities and also for describing the astrophysically interesting case of colour superconductivity [7].

The present review may to some extent be viewed as an extension of the material presented in the textbooks of Kapusta [8], Das [9] and LeBellac [10], with a certain amount of overlap with the latter which already covers the most important aspects of hard thermal loop resummations. Of course, numerous interesting new developments could only be touched upon, but hopefully enough references are provided as pointers to the recent literature; a lot of topics had to be omitted by lack of space or competence.

1.1. Outline

In section 2, after a brief recapitulation of the fundamental formulae of quantum statistical mechanics, the imaginary-time and real-time formalisms in thermal field theory are discussed with an overview of some more recently developed alternative approaches in the real-time formalism and in the treatment of gauge theories.

In section 3 we then exhibit some of the particular issues that arise in finite-temperature field theory in a simple scalar model: the appearance of thermal masses, the phenomenon of dimensional reduction at high temperatures, the need for a resummation of naive perturbation theory, and the problem of a deterioration

of apparent convergence after resummation and possibilities for its improvement. After a short discussion of the phenomenon of restoration of spontaneously broken symmetries at high temperatures at the end of section 3, section 4 describes the progress made during the last decade in the (resummed) perturbative evaluation of the thermodynamic potential of unbroken nonabelian gauge theory. At high temperature and not too high chemical potential, dimensional reduction provides the most efficient means for calculating perturbatively the thermodynamic potential, and this calculation has been carried to its limits, which are given by the inherent nonperturbative nature of nonabelian magnetostatic fields. It is argued that initial pessimism regarding the utility of weak-coupling methods in hot QCD is unnecessary, and that the results rather indicate that perturbative methods, when properly improved, may work already at temperatures that are only a few times higher than the deconfinement transition temperature. The recent progress made for the case of a nonvanishing fermion chemical potential is also reviewed, which includes quark number susceptibilities and non-Fermi-liquid contributions to the low-temperature entropy and specific heat of QCD and QED.

Section 5 discusses the structure of the propagators of a gauge theory with fermions, the leading-order results for the respective quasiparticles and the issue of gauge (in)dependence, both in unbroken nonabelian gauge theory and in the presence of colour superconductivity.

Section 6 then introduces the concept of hard-thermal/dense-loop resummation which is typically necessary (though not always sufficient) to calculate the effects of collective phenomena such as dynamical screening and propagating plasmons. Two recent approaches to improve the problem of the poor apparent convergence observed in the thermodynamic potential are discussed, which suggests that already the lowest order calculations in terms of quasiparticles may capture the most important contributions even in the strongly coupled quark-gluon plasma at temperatures that are only a few times higher than the deconfinement temperature.

The known results concerning next-to-leading order corrections to the quasiparticle spectrum in gauge theories are then reviewed in section 7. In this section a few simpler cases such as the leading infrared-sensitive contributions to the Debye mass and to damping rates and dynamical screening lengths are shown in more detail. Other cases of special interest are enhancements from collinear physics, which include non-Fermi-liquid corrections to the fermion self-energy in the vicinity of the Fermi surface and a modification of longitudinal plasmons for nearly light-like momenta.

Section 8 briefly discusses some of the progress made recently in the case of ultrasoft scales, where nonabelian gauge fields have nonperturbative dynamics, and in the case of observables that are sensitive to collinear physics, where also infinitely many loops contribute even after hard-thermal-loop resummation.

Section 9 finally considers thermal field theory in a curved background geometry and in particular the hard thermal loop contribution to the gravitational polarization tensor. The latter encodes the physics of cosmological perturbations in the presence of nearly collisionless ultrarelativistic matter, for which analytical solutions can be obtained in the physically relevant case of a conformally flat geometry.

2. Basics

In a relativistic quantum theory where interactions typically imply the destruction or creation of particles it is appropriate to formulate a statistical description by means

of the grand canonical ensemble.

A system in thermodynamical equilibrium for which only mean values of energy and any conserved charges are prescribed is characterized by a density matrix $\hat{\rho}$ such that $[\hat{\rho}, \hat{H}] = 0$ with $\hat{H}$ the Hamiltonian operator and the requirement of maximal entropy

$$S = \langle -\ln \hat{\rho} \rangle \equiv -\operatorname{Tr} \hat{\rho} \ln \hat{\rho}. \tag{2.1}$$

The temperature T and the chemical potentials μ_i appear as Lagrange multipliers $\beta = T^{-1}$ and $\alpha_i = -\beta\mu_i$ determining the mean energy $\langle \hat{H} \rangle$ and mean charges $\langle \hat{N}_i \rangle$, respectively, with $[\hat{H}, \hat{N}_i] = 0 = [\hat{N}_i, \hat{N}_j]$. This singles out

$$\hat{\rho} = Z^{-1} \exp\{-\beta \hat{H} - \sum_i \alpha_i \hat{N}_i\}, \tag{2.2}$$

where the normalization factor Z is the grand canonical partition function

$$Z(V, \beta, \mu_i) = \operatorname{Tr} \exp\{-\beta \hat{H} - \sum_i \alpha_i \hat{N}_i\}. \tag{2.3}$$

The partition function, Z, determines all of the other conventional thermodynamic (or rather thermo-static) quantities such as pressure, entropy, energy, and charge densities, which are denoted by P, $\mathcal{S}$, $\mathcal{E}$, and $\mathcal{N}$, respectively. In the thermodynamic infinite-volume limit ($V \to \infty$) these are given by

$$P = T \frac{\partial \ln Z}{\partial V} = \frac{T}{V} \ln Z, \tag{2.4}$$

$$\mathcal{S} \equiv S/V = \frac{\partial P}{\partial T}, \tag{2.5}$$

$$\mathcal{E} \equiv \langle \hat{H} \rangle / V = -\frac{1}{V} \frac{\partial \ln Z}{\partial \beta}, \tag{2.6}$$

$$\mathcal{N}_i \equiv \langle \hat{N}_i \rangle / V = \frac{\partial P}{\partial \mu_i}. \tag{2.7}$$

Combining (2.1) and (2.2) one obtains the Gibbs-Duhem relation in the form

$$\mathcal{S} = \langle -\ln \hat{\rho} \rangle = \frac{1}{V} \ln Z + \beta \mathcal{E} + \sum_i \alpha_i \mathcal{N}_i = \beta (P + \mathcal{E} - \sum_i \mu_i \mathcal{N}_i) \tag{2.8}$$

or

$$E = -PV + TS + \mu_i N_i, \tag{2.9}$$

which explains why P was introduced as the (thermodynamic) pressure. A priori, the hydrodynamic pressure, which is defined through the spatial components of the energy-momentum tensor through $\frac{1}{3}\langle T^{ii} \rangle$, is a separate object. In equilibrium, it can be identified with the thermodynamic one through scaling arguments [11], which however do not allow for the possibility of scale (or "trace") anomalies that occur in all quantum field theories with non-zero β-function (such as QCD). In [12] it has been shown recently that the very presence of the trace anomaly can be used to prove the equivalence of the two in equilibrium.

All these formulae pertain to the rest frame of the heat bath. A manifestly covariant formulation can be obtained by explicitly introducing the 4-velocity vector u^μ of this rest frame. In thermal equilibrium the energy momentum tensor takes the form

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + P(u^\mu u^\nu - \eta^{\mu\nu}), \tag{2.10}$$

where $\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$ is the Minkowski metric. Introducing the 4-vectors $\beta^\mu = \beta u^\mu$ and $j_i^\mu = n_i u_\mu$, the grand canonical partition function can be written covariantly as [13]

$$Z = \mathrm{Tr}\exp\int d\Sigma_\mu\{-\beta_\nu \hat{T}^{\mu\nu} - \sum_i \alpha_i \hat{j}_i^\mu\} \tag{2.11}$$

with reference to a hypersurface Σ with normal vector u.

It is of course more convenient to stick to the rest frame of the heat bath. Lorentz invariance appears to be broken then, but it is usually a trivial matter to switch to Lorentz covariant expressions using u^μ.

Information about the dynamics of a thermal system can be obtained by considering thermal expectation values of the generally time-dependent observables (in the Heisenberg picture). In linear response theory, the time evolution of small disturbances of an equilibrium system is determined by the correlation functions of pairs of observables [14].

For both the purpose of calculating the partition function Z and correlation functions there exist two equivalent but rather differently looking formalisms to set up perturbation theory. These correspond to the two most popular choices of a complex time path in the path integral formula

$$\langle \mathrm{T}_c \hat{\varphi}_1 \cdots \hat{\varphi}_n \rangle = \mathcal{N}\int \mathcal{D}\varphi\, \varphi_1 \cdots \varphi_n \exp \mathrm{i} \int_{\mathcal{C}} dt \int d^3x\, \mathcal{L}\,, \tag{2.12}$$

where T_c denotes contour ordering along the complex time path $\mathcal{C}$ from t_0 to $t_0 - \mathrm{i}\beta$ such that $t_i \in \mathcal{C}$, and $t_1 \succeq t_2 \succeq \cdots \succeq t_n$ with respect to a monotonically increasing contour parameter. $\mathcal{L}$ is the Lagrangian, which in the presence of a chemical potential $\mu \neq 0$ may be replaced by $\mathcal{L} \to \bar{\mathcal{L}} = \mathcal{L} + \mu\mathcal{N}$, provided $\mathcal{N}$ does not contain time derivatives. Analyticity requires that along the complex time path the imaginary part of t is monotonically decreasing [11]; in the limiting case of a constant imaginary part of t along (parts of) the contour, distributional quantities (generalized functions) arise.

Perturbation theory is set up in the usual fashion. Using the interaction-picture representation one can derive

$$\langle \mathrm{T}_c \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \frac{Z_0}{Z} \langle \mathrm{T}_c \mathcal{O}_1 \cdots \mathcal{O}_n\, \mathrm{e}^{\mathrm{i}\int_{\mathcal{C}} \mathcal{L}_I} \rangle_0, \tag{2.13}$$

where $\mathcal{L}_I$ is the interaction part of $\mathcal{L}$, and the correlators on the right-hand-side can be evaluated by a Wick(-Bloch-DeDominicis) theorem:

$$\langle \mathrm{T}_c \mathrm{e}^{\mathrm{i}\int_{\mathcal{C}} d^4x\, j\varphi} \rangle_0 = \exp\{-\frac{1}{2}\int_{\mathcal{C}}\int_{\mathcal{C}} d^4x\, d^4x'\, j(x) D^c(x-x') j(x')\}, \tag{2.14}$$

where D^c is the 2-point function and this is the only building block of Feynman graphs with an explicit T and μ dependence. It satisfies the KMS (Kubo-Martin-Schwinger) condition

$$D^c(t - \mathrm{i}\beta) = \pm \mathrm{e}^{-\mu\beta} D^c(t), \tag{2.15}$$

stating that $\mathrm{e}^{\mathrm{i}\mu t} D(t)$ is periodic (anti-periodic) for bosons (fermions).

A KMS condition can be formulated for all correlation functions in thermal equilibrium, and can in turn be viewed as a general criterion for equilibrium [15]. There exists also a relativistic version of the KMS condition [16, 17], which encodes the stronger analyticity requirements of relativistic quantum field theories. The relativistic spectrum condition $H \geq |\mathbf{P}|$, where $\mathbf{P}$ is the total three-momentum,

implies analyticity involving all space-time variables $x \to z \in \mathbb{C}^4$ in tube domains $|\operatorname{Im} \mathbf{z}| < \operatorname{Im} z_0 < \beta - |\operatorname{Im} \mathbf{z}|$, whereas the usual KMS condition corresponds to restricting this to $\operatorname{Im} \mathbf{z} = 0$.

2.1. Imaginary-time (Matsubara) formalism

The simplest possibility for choosing the complex time path in (2.12) is the straight line from t_0 to $t_0 - \mathrm{i}\beta$, which is named after Matsubara [18] who first formulated perturbation theory based on this contour. It is also referred to as imaginary-time formalism (ITF), because for $t_0 = 0$ one is exclusively dealing with imaginary times.

Because of the (quasi-)periodicity (2.15), the propagator is given by a Fourier series

$$D^c(t) = \frac{1}{-\mathrm{i}\beta} \sum_\nu \tilde{D}(z_\nu) \mathrm{e}^{-\mathrm{i}z_\nu t}, \quad \tilde{D}(z_\nu) = \int_0^{-\mathrm{i}\beta} dt \, D^c(t) \mathrm{e}^{\mathrm{i}z_\nu t} \tag{2.16}$$

with discrete complex (Matsubara) frequencies

$$z_\nu = 2\pi\mathrm{i}\nu/\beta + \mu, \quad \nu \in \begin{cases} \mathbb{Z} & \text{bos.} \\ \mathbb{Z} - \frac{1}{2} & \text{ferm.} \end{cases} \tag{2.17}$$

Since $\tilde{D}(z_\nu)$ is defined only for a discrete set of complex number, the analytic continuation to arbitrary frequencies is unique only when one requires that $|\tilde{D}(z)| \to 0$ for $|z| \to \infty$ and that $\tilde{D}(z)$ is analytic off the real axis [19]. Then the analytic continuation is provided by the spectral representation

$$\tilde{D}(z) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{\rho(k_0)}{k_0 - z}. \tag{2.18}$$

The transition to Fourier space turns the integrands of Feynman diagrams from convolutions to products as usually, with the difference that there is no longer an integral but a discrete sum over the frequencies, and compared to standard momentum-space Feynman rules one has

$$\int \frac{d^4k}{\mathrm{i}(2\pi)^4} \to \beta^{-1} \sum_\nu \int \frac{d^3k}{(2\pi)^3}, \qquad \mathrm{i}(2\pi)^4 \delta^4(k) \to \beta(2\pi)^3 \delta_{\nu,0} \delta^3(k). \tag{2.19}$$

However, all Green functions that one can calculate in this formalism are initially defined only for times on $\mathcal{C}$, so all time arguments have the same real part. The analytic continuation to several different times on the real axis is, however, frequently a highly involved task [11], so that it can be advantageous to use a formalism that supports real time arguments from the start.

2.2. Real-time (Keldysh) formalism

In the so-called real-time formalism(s), the complex time path $\mathcal{C}$ is chosen such as to include the real-time axis from an initial time t_0 to a final time t_f. This requires further parts of the contour to run backward in real time [20, 21] and to end up at $t_0 - \mathrm{i}\beta$. There are a couple of paths $\mathcal{C}$ that have been proposed in the literature. The oldest one due to Keldysh [22] is shown in figure 1, where the first part of the contour $\mathcal{C}_1$ is on the real axis, and a second part runs from $t_f - i\delta$ to $t_0 - i\delta$ with $\delta \to 0$. For some time, the more symmetric choice where the backward-running contour is placed such as to have imaginary part $-\delta = -\beta/2$ with two vertical contour pieces at t_0 and t_f of equal length has enjoyed some popularity [23, 24, 11] in particular in

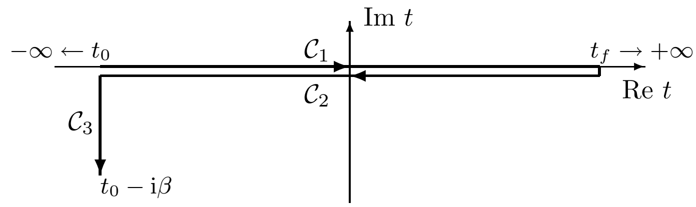


Figure 1. Complex time path in the Schwinger-Keldysh real-time formalism

the axiomatic thermo-field-dynamics (TFD) operator formalism [25, 26, 27, 28] but by now the original Keldysh contour seems to be the one most widely employed, above all because of its close relationship to the so-called closed time-path formalism [22, 29, 30] of nonequilibrium thermodynamics.

With the Keldysh contour, if none of the field operators in (2.12) has time argument on $\mathcal{C}_1$ or $\mathcal{C}_2$, the contributions from these parts of the contour simply cancel and one is back to the ITF. On the other hand, if all operators have finite real time arguments, the contribution from contour $\mathcal{C}_3$ decouples. The standard argument for this assertion [23, 24, 11] relies on the limit $t_0 \to -\infty$ and the fact that the propagator connecting contour $\mathcal{C}_1$ and $\mathcal{C}_3$ decays by virtue of the Riemann-Lebesgue theorem

$$D^{13}(t-(t_0-\mathrm{i}\lambda),k) = \int_{-\infty}^{\infty} d\omega\, \mathrm{e}^{-\mathrm{i}\omega(t-t_0)} \frac{\mathrm{e}^{\lambda\omega}}{\mathrm{e}^{\beta\omega}-1} \rho(\omega,k) \overset{t_0\to-\infty}{\longrightarrow} 0 \tag{2.20}$$

for $\lambda \in (0,\beta)$. However, as argued by Niégawa[31] there are cases where this line of reasoning breaks down. The decoupling of the vertical part(s) of the contour in RTF does however take place provided the statistical distribution function in the free RTF propagator defined below in (2.23) does have as its argument $|k_0|$ and not the seemingly equivalent $\omega_k = (\mathbf{k}^2+m^2)^{1/2}$. Reference [31] also relied on the limit $t_0 \to -\infty$, but Gelis [32, 33] pointed out that the decoupling of $\mathcal{C}_3$ should take place regardless of the magnitude of t_0, and showed this indeed to be the case provided the "$n(|k_0|)$ prescription" is used (see also [34, 35]).

With only $\mathcal{C}_1$ and $\mathcal{C}_2$ contributing, the action in the path integral decomposes according to

$$\int_{\mathcal{C}_1\cup\mathcal{C}_2} \mathcal{L}(\varphi) = \int_{-\infty}^{\infty} dt\, \mathcal{L}(\varphi^{(1)}) - \int_{-\infty}^{\infty} dt\, \mathcal{L}(\varphi^{(2)}) \tag{2.21}$$

where one has to distinguish between fields of type 1 (those from contour $\mathcal{C}_1$) and of type 2 (those from contour $\mathcal{C}_2$) because of the prescription of contour ordering in (2.12).‡ From (2.21) it follows that type-1 fields have vertices only among themselves, and the same holds true for the type-2 fields. However, the two types of fields are coupled through the propagator, which is a 2×2 matrix with non-vanishing off-diagonal elements:

$$\mathbf{D}^c(t,t') = \begin{pmatrix} \langle \mathrm{T}\varphi(t)\varphi(t')\rangle & \sigma\langle\varphi(t')\varphi(t)\rangle \\ \langle\varphi(t)\varphi(t')\rangle & \langle\tilde{\mathrm{T}}\varphi(t)\varphi(t')\rangle \end{pmatrix}. \tag{2.22}$$

Here $\tilde{\mathrm{T}}$ denotes anti-time-ordering for the 2-2 propagator and σ is a sign which is positive for bosons and negative for fermions. The off-diagonal elements do not need

‡ Type-2 fields are sometimes called "thermal ghosts", which misleadingly suggests that type-1 fields are physical and type-2 fields unphysical. In fact, they differ only with respect to the time-ordering prescriptions they give rise to.

a time-ordering symbol because type-2 is by definition always later (on the contour) than type-1.

In particular, for a massive scalar field one obtains

$$\begin{aligned}\mathbf{D}^c(k) &= \begin{pmatrix} \frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\epsilon} & 2\pi\delta^-(k^2-m^2) \\ 2\pi\delta^+(k^2-m^2) & \frac{-\mathrm{i}}{k^2-m^2-\mathrm{i}\epsilon} \end{pmatrix} + 2\pi\delta(k^2-m^2)n(|k_0|)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ &\equiv \mathbf{D}^c_0(k) + \mathbf{D}^c_T(k) \end{aligned} \tag{2.23}$$

where $n(\omega) = [\mathrm{e}^{\beta\omega}-1]^{-1}$ and $\delta^\pm(k^2-m^2) = \theta(\pm k_0)\delta(k^2-m^2)$. The specifically thermal contribution $\mathbf{D}^c_T$ is a homogeneous Green function, as it is proportional to $\delta(k^2-m^2)$. Physically, this part corresponds to Bose-Einstein-distributed, real particles on mass-shell.

The matrix propagator (2.23) can also be written in a diagonalized form [36, 28, 37]

$$\mathbf{D}^c(k) = \mathbf{M}(k_0)\begin{pmatrix} \mathrm{i}G_F & 0 \\ 0 & -\mathrm{i}G_F^* \end{pmatrix}\mathbf{M}(k_0) \tag{2.24}$$

with $G_F \equiv 1/(k^2-m^2+\mathrm{i}\epsilon)$ and

$$\mathbf{M}(k_0) = \frac{1}{\sqrt{\mathrm{e}^{\beta|k_0|}-1}}\begin{pmatrix} \mathrm{e}^{\frac12\beta|k_0|} & \mathrm{e}^{(\delta-\frac12\beta)k_0} \\ \mathrm{e}^{(\frac12\beta-\delta)k_0} & \mathrm{e}^{\frac12\beta|k_0|} \end{pmatrix} \tag{2.25}$$

where $\delta \to 0$ for the Keldysh contour.

For complex time paths with $\delta \propto \beta$, the $T \to 0$ limit decouples type-1 and type-2 fields completely. For $\delta \to 0$, the limit $T \to 0$ leads to

$$\mathbf{M}(k_0) \stackrel{\beta\to\infty}{\longrightarrow} \mathbf{M}_0(k_0) = \begin{pmatrix} 1 & \theta(-k_0) \\ \theta(k_0) & 1 \end{pmatrix} \tag{2.26}$$

so that one still has propagators connecting fields of different type. However, if all the external lines of a diagram are of the same type, then also all the internal lines are, because $\prod_i \theta(k^0_{(i)}) = 0$ when $\sum_i k^0_{(i)} = 0$ and any connected region of the other field-type leads to a factor of zero.

The matrix structure (2.24) also applies to the full propagator, and consequently the self energy $\mathrm{i}\mathbf{\Pi} = \mathbf{D}^{-1} - \mathbf{D}_0^{-1}$ has the analogous form with $\mathbf{M}^{-1}$ in place of $\mathbf{M}$,

$$\mathbf{\Pi}(k) = \mathbf{M}(k_0)\begin{pmatrix} \Pi_F & 0 \\ 0 & -\Pi_F^* \end{pmatrix}\mathbf{M}(k_0). \tag{2.27}$$

It is also possible to diagonalize in terms of retarded and advanced quantities according to

$$\mathbf{D}^c(k) = \mathbf{U}(k_0)\begin{pmatrix} \mathrm{i}G_R & 0 \\ 0 & \mathrm{i}G_A^* \end{pmatrix}\mathbf{V}(k_0) \tag{2.28}$$

with

$$\mathbf{U}(k_0) = \begin{pmatrix} 1 & -n(k_0) \\ 1 & -(1+n(k_0)) \end{pmatrix}, \quad \mathbf{V}(k_0) = \begin{pmatrix} 1+n(k_0) & n(k_0) \\ 1 & 1 \end{pmatrix} \tag{2.29}$$

and to include the matrices $\mathbf{U}$ and $\mathbf{V}$ in the vertices [38, 39]. This has the advantage of leading to n-point Green functions with well-defined causal properties, which correspond directly to the various analytic continuations of ITF Green functions. On the other hand, the type-1/type-2 basis leads to rather involved relations [40, 41, 42].

Because of

$$\mathbf{U}(k)\tau_1 = \mathbf{V}^T(-k), \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.30}$$

more symmetric retarded/advanced Feynman rules can be formulated by including a factor τ_1 in $\mathbf{U}$ and putting G_R and G_A in the off-diagonal entries of (2.28) [43, 44].

A precursor of this transformation is in fact given by the so-called Keldysh basis [22, 29, 44]

$$\varphi_+ = \tfrac{1}{2}(\varphi_1 + \varphi_2), \quad \varphi_- = (\varphi_1 - \varphi_2) \tag{2.31}$$

(sometimes labelled by indices r, a instead). This also has the advantage of a rather direct relationship to retarded/advanced n-point Green functions, and because the transformation does not involve $n(k_0)$, it is of use also in the nonequilibrium closed-time-path formalism [45].

Another economical method to derive retarded/advanced quantities in the real time formalism is provided by the use of the outer products of 2-component column vectors [46, 47, 48] as worked out in [49, 50].

2.3. Extension to gauge theories

In the partition function (2.3) and in thermal averages $\langle\hat{\mathcal{Q}}\rangle \equiv \mathrm{Tr}\,\hat\rho\hat{\mathcal{Q}}$, the trace is taken over the physical Hilbert space. But covariant formulations of gauge theories are built in larger spaces containing unphysical states, while the definitions following (2.3) are true only in the physical Hilbert subspace. The standard solution is to extend the trace to the larger unphysical space and to cancel unphysical contributions by Faddeev-Popov ghosts.

In the path integral formalism, the Faddeev-Popov ghost fields arise from a functional determinant in the configuration space of the bosonic gauge fields. This requires that although Faddeev-Popov fields behave as fermions with respect to the diagrammatical combinatorics, they are subject to the same boundary conditions as the gauge bosons and therefore have the same statistical distribution functions, namely Bose-Einstein ones [51].

In operator language which starts from a BRS invariant theory involving fermionic Faddeev-Popov field operators [52], this prescription can be understood through the observation [53] that the operator $\exp i\pi\hat{N}_c$, where $\hat{N}_c$ is the ghost-number operator, is equivalent, under the trace, to a projection operator onto the physical Hilbert space. This means that the fermionic Faddeev-Popov fields are given an imaginary chemical potential $\mu_c = i\pi/\beta$. But a Fermi-Dirac distribution with such a chemical potential is nothing other than a Bose-Einstein distribution.

There is however an alternative approach, developed in [54, 55], which avoids assigning thermal distributions to unphysical degrees of freedom altogether. In the real-time formalism, one may switch off the gauge coupling adiabatically as the beginning of the time contour is moved to $-\infty$. Then the condition to choose physical states can be the same as in Abelian gauge theory. The unphysical states are identified as those which are due to the Faddeev-Popov fields and the temporal and longitudinal polarizations of the gauge fields. Because the free Hamiltonian is a sum of commuting parts containing respectively only physical and unphysical operators, and because the unphysical part has zero eigenvalue on the physical states, all unphysical contributions factor out such that only the transverse polarizations of the gauge fields acquire thermal parts in their propagators.

In Feynman gauge for instance the gauge propagator, which usually is simply $\mathbf{D}^{\mu\nu} = -\eta^{\mu\nu}\mathbf{D}^c$, with $\mathbf{D}^c$ the matrix propagator (2.23), becomes

$$\mathbf{D}^{\mu\nu} = -A^{\mu\nu}\mathbf{D}^c - (\eta^{\mu\nu} - A^{\mu\nu})\mathbf{D}^c_0 \tag{2.32}$$

with

$$A^{0\mu} = 0, \quad A^{ij} = -(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2}) \tag{2.33}$$

and the Faddeev-Popov ghost propagator remains non-thermal, $\mathbf{D}_{\mathrm{FP}} = \mathbf{D}_0^c$.

In a general linear gauge with a quadratic gauge breaking term in momentum space according to

$$\tilde{\mathcal{L}}_{\text{g.br.}} = -\frac{1}{2\alpha}\tilde{A}^\mu(-k)\tilde{f}_\mu \tilde{f}_\nu \tilde{A}^\nu(k) \tag{2.34}$$

the vacuum piece generalizes by replacing

$$\eta^{\mu\nu} \to \eta^{\mu\nu} - \frac{k^\mu \tilde{f}^\nu + \tilde{f}^\mu k^\nu}{\tilde{f}\cdot k} + (\tilde{f}^2 + \alpha k^2)\frac{k^\mu k^\nu}{(\tilde{f}\cdot k)^2}. \tag{2.35}$$

The ghost propagator is replaced by $(\tilde{f}\cdot k)^{-1}$ with real-time propagator matrix analogous to the vacuum part of the gauge propagator.

At finite temperature, where manifest Lorentz invariance has been lost anyway, the modification (2.32) does not introduce additional non-covariance. In fact, it simplifies calculations of thermal contributions in general gauges [54], but it makes it more intricate to investigate resummation effects [55].

In gauges where the ghost degrees of freedom are non-thermal anyway, such as Coulomb gauge or axial gauges, the above Feynman rules are identical to those of the conventional approach. In particular, it reproduces the real-time Feynman rules for temporal axial gauge of [56] which presents major difficulties in the imaginary-time formalism [57].

3. Resummation issues in scalar ϕ^4-theory

Before discussing gauge theories further, we shall consider perturbation theory at finite temperature in a scalar field theory with quartic coupling and address the necessity for resummations of the perturbative series.

3.1. Daisy and foam resummation

A particularly simple "solvable" model is given by the the large N limit of a massless O(N) scalar field theory with quartic interactions as given by the Lagrangian [58, 59, 60, 61, 62]

$$\mathcal{L}(x) = \tfrac{1}{2}(\partial_\mu \boldsymbol{\phi})^2 - \frac{3}{N+2}g_0^2(\boldsymbol{\phi}^2)^2 \tag{3.1}$$

where there are N scalar fields ϕ_i and $\boldsymbol{\phi}^2 = \phi_1^2 + \ldots \phi_N^2$.

In the limit of $N \to \infty$ the only Feynman diagrams that survive are those that derive from ring ("daisy") diagrams or nested rings ("superdaisies", "cactus", or "foam" diagrams) as shown in figure 2.

In dimensional regularization, the zero-temperature scalar theory is massless provided the bare mass is zero. The coupling however receives an infinite renormalization in $n \to 4$ dimensions. In the large-N limit this is determined in modified minimal subtraction ($\overline{\mathrm{MS}}$) by

$$\frac{1}{g^2} = \frac{1}{g_0^2} + \frac{3}{2\pi^2}\frac{1}{4-n}\,. \tag{3.2}$$

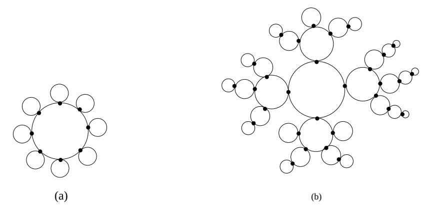


Figure 2. Ring and extended ring or "foam" diagrams

The sign of the counterterm in (3.2) in fact hints at the well-known problem of triviality of ϕ^4 theories [61]. If the bare coupling g_0^2 is positive and n approaches 4 from below, the renormalized coupling g goes to zero. As discussed in reference [62], if one insists on a non-trivial theory with $g^2 > 0$ (which is only possible when g_0^2 is negative, and divergent for $n \to 4$), one finds that there is a tachyon (Landau pole) with mass

$$m_{\rm tachyon}^2 = -\bar{\mu}^2 \exp\left(\frac{8\pi^2}{3g^2} + 2\right) \tag{3.3}$$

which appears to disqualify this model completely. Here $\bar{\mu}^2 = 4\pi e^{-\gamma}\mu^2$ and μ is a mass scale introduced to make the coupling dimensionless in $n \neq 4$ dimensions.

However, at small renormalized coupling, $g^2 \ll 1$, the tachyon's mass is exponentially large. If everything is restricted to momentum scales smaller than (3.3), e.g. by a slightly smaller but still exponentially large cutoff, the $n=4$ theory seems perfectly acceptable. For our purposes we shall just have to restrict ourselves to temperature scales smaller than (3.3) when considering the finite-temperature effects in this scalar theory.

3.2. Thermal masses

To one-loop order, the scalar self-energy diagram is the simple tadpole shown in the first diagram of figure 3, which is quadratically divergent in cutoff regularization, but strictly zero in dimensional regularization. The Bose distribution function occurring at nonzero temperature provides a cutoff at the scale of the temperature which gives

$$\Pi = (m_{\rm th}^{(1)})^2 = 4!\, g^2 \int \frac{d^3q}{(2\pi)^3} \theta(q^0)\delta(q^{02} - \boldsymbol{q}^2)\{n(q^0) + \tfrac{1}{2}\} = g^2 T^2 \tag{3.4}$$

so the initially massless scalar fields acquire a temperature-dependent mass. As we shall see, in more complicated theories the thermal self-energy will generally be a complicated function of frequencies and momenta, but the appearance of a thermal mass scale $\sim gT$ is generic.

It should be noted, however, that thermal masses are qualitatively different from ordinary Lorentz-invariant mass terms. In particular they do not contribute to the trace of the energy-momentum tensor as m^2T^2, as an ordinary zero-temperature mass would do [63]. So while the dispersion law of excitations is changed by the thermal medium, the theory itself retains its massless nature.

At higher orders in perturbation theory, the thermal contributions to the scalar self-energy become nontrivial functions of frequency and momentum which is complex-valued, implying a finite but parametrically small width of thermal (quasi) particles [64, 65], so that the latter concept makes sense perturbatively.

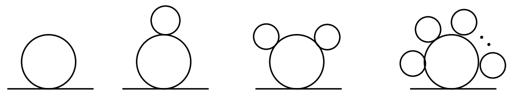


Figure 3. One-loop correction to the self-energy in scalar ϕ^4 theory, and some-higher-loop diagrams with increasing degree of infrared singularity when the propagators are massless

In the large-N limit the self-energy of the scalar field remains a momentum-independent real mass term also beyond one-loop order and is given by the Dyson equation

$$\Pi = 4!\, g_0^2\, \mu^{4-n} \int \frac{d^n q}{(2\pi)^{n-1}} \{n(q^0) + \tfrac{1}{2}\} \theta(q^0) \delta(q^2 - \Pi). \tag{3.5}$$

(Note that in (3.4) we had simply replaced g_0^2 by g^2, disregarding the difference as being of higher than one-loop order.) The appearance of a thermal mass introduces quadratic ultraviolet divergences in Π, which are however of exactly the form required by the renormalization of the coupling according to (3.2). Including the latter, one finds a closed equation for the thermal mass of the form

$$\begin{aligned} m_{\rm th}^2 &= 24g^2 \Big\{ \int \frac{d^n q}{(2\pi)^{n-1}} n(q^0)\theta(q^0)\delta(q^2 - m_{\rm th}^2) + \frac{1}{32\pi^2} m_{\rm th}^2 \left(\ln \frac{m_{\rm th}^2}{\bar\mu^2} - 1 \right) \Big\} \\ &=: 24g^2 \left\{ I_T(m_{\rm th}) + I_0^f(m_{\rm th}, \bar\mu) \right\}. \end{aligned} \tag{3.6}$$

The last term in the braces in (3.6), which has been neglected in [58, 66, 10], is responsible for a non-trivial interplay between thermal and vacuum contributions. Its explicit dependence on $\bar\mu$ is such that it cancels the one implicit in $g^2 = g^2(\bar\mu)$,

$$\frac{dg}{d\ln(\bar\mu)} = \frac{3}{4\pi^2} g^3, \tag{3.7}$$

which is exact for $N \to \infty$.

More complicated effects of zero-temperature renormalization on reorganized thermal perturbation theories have been discussed in a scalar ϕ_6^3 model in [67]. Another solvable toy model is given by the large-N_f limit of QCD or QED, which has been worked out in a thermal field theory context in [68, 69, 70, 71]. These theories also require the introduction of a cutoff to avoid the Landau singularity and triviality. In contrast to the $\mathrm{O}(N \to \infty)$ ϕ^4 theory, they involve complicated momentum-dependent dispersion laws as well as damping effects. While the large-N ϕ_6^3 theory has unphysical instabilities above a certain temperature (aside from the Landau singularity), large-N_f QED and QCD is well-behaved and very useful as a benchmark for approximations to real QED and QCD.

3.3. Perturbation series

The integral appearing in (3.6) can be evaluated e. g. using Mellin transformation techniques [72] to obtain a series expansion of $m_{\rm th}$ whose first few terms read [62]

$$\frac{m_{\rm th}^2}{T^2} = g^2 - \frac{3g^3}{\pi} + \frac{3g^4}{2\pi^2} \left(3 - \gamma - \ln \frac{\bar\mu}{4\pi T} \right) + \frac{27g^5}{8\pi^3} \left(2\gamma + 2\ln \frac{\bar\mu}{4\pi T} - 1 \right) + \mathrm{O}(g^6). \tag{3.8}$$

This result shows that ordinary perturbation theory is unable to go beyond the one-loop result (3.4), for ordinary perturbation theory is an expansion in powers of g^2.

Equation (3.8) however involves odd powers of g. Indeed, because of the masslessness of the scalar theory, ordinary perturbation theory encounters infrared divergences starting at two-loop order, which are exacerbated by the Bose distribution function behaving as $n(q^0) \sim T/q^0$ for $q^0 \to 0$. For example, the second diagram in figure 3 involves two massless propagators with equal momentum. The inserted tadpole does not vanish at zero momentum, but is given by the constant term (3.4). At zero temperature, such an insertion (if nonzero), would make this diagram logarithmically infrared divergent; at finite temperature it is instead linearly infrared divergent. The higher-loop diagrams shown in figure 3 are even more infrared divergent. But, as we have seen, the full propagator contains the thermal mass, so all these divergences are spurious—they just signal the need for using a resummed i.e. massive propagator.

A systematic method to perform the required resummation of ordinary perturbation theory is to add a thermal mass term $-1/2 m_{\rm th}^2 \boldsymbol{\phi}^2$ to the Lagrangian (3.1) and to subtract it as a counter-term which is treated as a one-loop-order quantity. The corresponding calculation at $N = 1$, where (3.6) is only part of the full result, has been performed up to and including order g^5 in [64, 73].

3.4. Dimensional reduction

An important technical concept for studying static quantities such as thermodynamic potentials and (static) screening masses is that of "dimensional reduction" [74, 75, 76, 77, 78, 79, 80], which in the case of scalar field theories has been worked out in [81]. In this approach one separates hard ($k \sim T$) from soft ($k \lesssim gT$) modes and integrates out the former. Since in the Matsubara formalism all non-static modes are necessarily hard, this yields a three-dimensional effective theory containing zero-modes only whose parameters (masses and coupling constants) are to be determined by perturbative matching to the full theory. To lowest order in scalar ϕ^4 theory, this yields (now for $N = 1$)

$$\mathcal{L}_3 = \tfrac{1}{2}(\nabla\phi_0)^2 + \tfrac{1}{2}m_3^2\phi_0^2 + g_3^2\phi_0^4 + \ldots \tag{3.9}$$

where $\phi_0 = \sqrt{T}\int_0^\beta d\tau\phi(\tau, \mathbf{x})$ and, to lowest order, $m_3^2 = g^2T^2$, $g_3^2 = g^2T$.

Calculating now also soft one-loop corrections, one obtains in dimensional regularization

$$\delta m^2 = 12 g_3^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2 + m_3^2} = -\frac{3}{\pi}g_3^2 m_3 = -\frac{3}{\pi}g^3T^2, \tag{3.10}$$

which is exactly the correction given in (3.8). (In cutoff regularization there would be an extra term $\propto g_3^2\Lambda$, cancelling a contribution $\propto g^2T\Lambda$ to m_3^2.) Using this method, the perturbative expansion of m_{th} for $N = 1$ has been worked out to order g^5 in [73].

3.5. Apparent convergence

The large-N limit provides an instructive opportunity to study the convergence properties of a resummed perturbation theory with the exact result obtained by simply solving (3.6) numerically. This has been carried out in great detail in [62] with the result that only for g sufficiently smaller than 1 there is quick convergence, which further deteriorates if the renormalization scale $\bar{\mu}$ is very different from T. An optimal value turned out to be $\bar{\mu} \approx 2\pi T$, the scale of the bosonic Matsubara frequencies, which has been argued previously in [81] to be a natural choice.

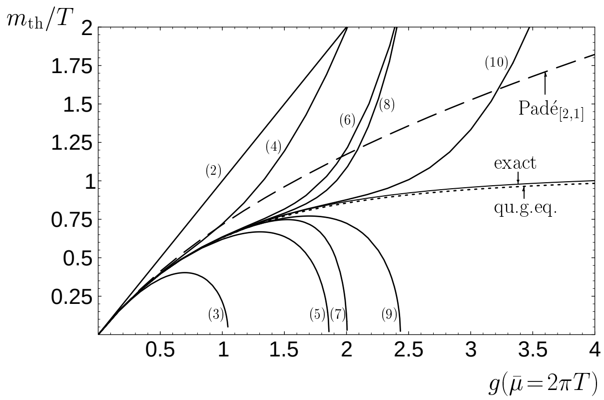


Figure 4. Thermal mass in large-N ϕ^4-theory as a function of $g(\bar\mu = 2\pi T)$ together with the perturbative results (3.8) accurate to order g^2, g^3, …, g^{10}. The g^3 result is the one reaching zero at $g \approx 1$; its Padé-improved version (3.12) is given by the long-dashed line. The short-dashed line just below the exact result is obtained by solving the quadratic gap equation (3.13), which is also perturbatively equivalent to the order g^3 result.

However, the convergence of the resummed thermal perturbation series seems to be surprisingly poor given that the exact result following from (3.6) is a rather unspectacular function. Naturally, a perturbative result is a truncated polynomial and thus bound to diverge more and more rapidly at large coupling as the order is increased. This may be the case even when the physical effects described by the lowest-order terms are still predominant.

In [82] it has been shown that the alternative so-called nonlinear δ-expansion scheme [83] yields approximations to $m^2_{\rm th}$ that converge almost uniformly in g, but this scheme has not yet found applications in more complicated (gauge) field theories.

A simpler alternative is provided by Padé approximants [84], that is rewriting a given perturbative result as a perturbatively equivalent rational function in g by replacing

$$F_n(g) = c_0 + c_1 g^1 + \ldots + c_n g^n \to F_{[p,n-p]}(g) = \frac{c_0 + a_1 g^1 + \ldots + a_p g^p}{1 + a_{p+1} g^1 + \ldots + a_n g^{n-p}}. \tag{3.11}$$

For example, truncating (3.8) above order g^3 gives an approximation F_3 to $m^2_{\rm th}$ that stops growing as a function of g at $g = 2\pi/9 \approx 0.7$ and goes back to zero and then to negative values for $g \geq \pi/3 \approx 1$ (line labelled "(3)" in figure 4). The true (large-N) thermal mass from (3.6) however is a monotonically growing function (line labelled "exact" in figure 4). The result F_3 in fact ceases to be an improvement over the leading-order result $F_2 = g^2 T^2$ roughly where it stops growing. On the other hand, the simplest possibility for a perturbatively equivalent Padé approximant, $F_{[2,1]}$, is a monotonic function in g,

$$m^2_{\rm th}/T^2 = \frac{g^2}{1 + 3g/\pi} + {\rm O}(g^4) \tag{3.12}$$

(long-dashed line in figure 4), and it gives a substantial improvement for $g \gtrsim 1$.

Higher-order Padé approximants converge rather well in the simple scalar model [62] (except when they happen to have a pole of the denominator at positive coupling)

and they have been proposed as a possibility to improve also the unsatisfying convergence of perturbative results for the thermodynamical potential in finite-temperature QCD [85, 86, 62], which will be discussed in more detail in Sect. 4.2. There it does increase the apparent convergence of the resummed perturbation series for the first few orders, but at higher orders it looks less convincing. In fact, at these higher orders the perturbation series also involves $\ln(g)$-contributions, which make a simple Padé improvement appear less natural. Indeed, in the above large-N scalar model, where the higher Padé approximants converge rather quickly [62], no $\ln(g)$-terms arise.

Already the simplest Padé resummation (3.12) suggests that the low quality of standard perturbative results is due to the fact that the latter are polynomials in g which inevitably blow up at larger values of g. In particular those contributions which can be traced to a resummation of the screening mass involve large coefficients. Since this resummation is a priori nonperturbative in that it involves arbitrarily high powers of g, this signals the need for a more complete treatment of such resummation effects.

In the above scalar toy model, one can in fact easily obtain an efficient resummation of the term involving g^3 in (3.8) which is responsible for the poor convergence of the perturbative results displayed in figure 4. If one just retains the first two terms in a (m/T) expansion of the one-loop gap equation (3.6), one ends up with a simple quadratic gap equation

$$m_{\rm th}^2 = g^2T^2 - \frac{3}{\pi} g^2 T m_{\rm th} \tag{3.13}$$

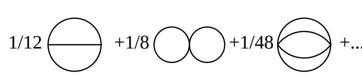
which is perturbatively equivalent to (3.8) to order g^3, but the solution of (3.13) turns out to be extremely close to that of the full gap equation if the $\overline{\rm MS}$ renormalization scales $\bar\mu \approx 2\pi T$.

There exist formalisms which at a given order of approximations perform a complete propagator resummation: the so-called self-consistent Φ-derivable approximations [87] (in particle physics also known in connection with the composite-operator effective action or Cornwall-Jackiw-Tomboulis formalism [88]).

In the Luttinger-Ward representation [89] the thermodynamic potential $\Omega = -PV$ is expressed as a functional of full propagators D and two-particle irreducible (2PI) diagrams. Considering a scalar field theory with both cubic and quartic vertices for the moment, $\Omega[D]$ has the form

$$\begin{aligned}\Omega[D] =& -T\log Z = \tfrac{1}{2}T\ {\rm Tr}\log D^{-1} - \tfrac{1}{2}T\ {\rm Tr}\ \Pi D + T\Phi[D] \\ =& \int \frac{d^4k}{(2\pi)^4} n(\omega)\, {\rm Im}\ \left[\log D^{-1}(\omega,k) - \Pi(\omega,k)D(\omega,k)\right] + T\Phi[D],\end{aligned} \tag{3.14}$$

where Tr denotes the trace in configuration space, and $\Phi[D]$ is the sum of the 2-particle-irreducible "skeleton" diagrams

$-\Phi[D] =$  (3.15)

The self energy $\Pi = D^{-1} - D_0^{-1}$, where D_0 is the bare propagator, is related to $\Phi[D]$ by

$$\delta\Phi[D]/\delta D = \tfrac{1}{2}\Pi. \tag{3.16}$$

An important property of the functional $\Omega[D]$, which is easily verified using (3.16), is that it is stationary under variations of D:

$$\delta\Omega[D]/\delta D = 0. \tag{3.17}$$

Self-consistent ("Φ-derivable") [87] approximations are obtained by selecting a class of skeletons in $\Phi[D]$ and calculating Π from equation (3.16) above, preserving the stationarity condition.

Φ-derivable approximations have been worked out in scalar theory to 3-loop order [90, 91] and it has recently been shown that in this model these approximations can be nonperturbatively renormalized in a self-consistent manner [92, 93, 94, 95].

In the present large-N ϕ^4 toy model, the 2-loop Φ-derivable approximation is in fact exact, so that one may use this example for studying the quality of further approximations on top of the former [5], which are typically unavoidable to make the formalism tractable in more complicated theories.

In particular, a resummation of quasi-particle propagators in thermodynamic quantities is conveniently performed to 2-loop order in the entropy density $\mathcal{S} = dP/dT$, which turns out to have the simple representation [96]

$$\mathcal{S} = -\int \frac{d^4k}{(2\pi)^4} \frac{\partial n(\omega)}{\partial T} \left\{ \mathrm{Im}\, \log D^{-1}(\omega, k) - \mathrm{Im}\, \Pi(\omega, k)\, \mathrm{Re}\, D(\omega, k) \right\}, \tag{3.18}$$

where Π is the one-loop self-energy calculated self-consistently with dressed propagator D. At this order all fundamental interactions can be completely absorbed in the spectral properties of the quasi-particles, whose residual interactions enter only at three-loop order. For a real momentum-independent self-energy $\Pi = m^2$, (3.18) even coincides with the free (Stefan-Boltzmann) expression for the entropy density, $\mathcal{S}_{\mathrm{free}}(m)$. The more general form (3.18) is also applicable in gauge theories and for fermions [97, 96, 98, 99] and we shall return to that in Sect. 6.3.2.

In references [100, 101, 102, 103] a different reorganization of thermal perturbation theory has been proposed, "screened perturbation theory" (SPT), which amounts to adding and subtracting a mass term to the Lagrangian (3.1) according to

$$\mathcal{L} \to \mathcal{L} - \tfrac{1}{2} m^2 \boldsymbol{\phi}^2 + \delta \tfrac{1}{2} m^2 \boldsymbol{\phi}^2 \tag{3.19}$$

and to treating δ as a one-loop quantity prior to putting it eventually to $\delta = 1$. The difference to conventional resummation techniques is to refrain from identifying and expanding out the coupling g implicit in m, which in the end is chosen as some approximation to the thermal mass and thus proportional to g for small coupling.

Starting from two-loop order, it is possible to determine m by a principle of minimal sensitivity, which makes SPT a variant of the so-called variational perturbation theories (see e.g. [104, 105]).

This scheme has been applied with apparent success to scalar field theory [100, 101, 102, 103]. It has also been generalized to gauge theories, where a local mass term is insufficient but needs to be replaced by a nonlocal gauge invariant extension of a thermal mass term (see sect. 6.3.1) [106, 107, 108, 109]. A special difficulty of SPT is that at any finite order of the new perturbative expansion it gives rise to new ultra-violet divergences and corresponding new scheme dependences, which need to be fixed in some way or other [110].

3.6. Restoration of spontaneously broken symmetry

A physically important case ignored so far is spontaneous symmetry breaking, the simplest example of which is provided by adding a wrong-sign mass term to the ϕ^4 theory considered above

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - V_{\mathrm{cl}}(\phi) \quad \text{with } V_{\mathrm{cl}}(\phi) = -\tfrac{1}{2}\nu^2 + \tfrac{\lambda}{4!}\phi^4, \tag{3.20}$$

where we have switched to the more conventional notation of $\lambda/4! = g^2$. While $\mathcal{L}$ is symmetric under $\phi \to -\phi$, this symmetry is "spontaneously broken" by choosing one of the minima of $V_{\rm cl}$, which are given by $\phi_{\rm min} = \pm\sqrt{3!\nu^2/\lambda}$.

At high temperatures, however, there is symmetry restoration [111, 112, 58, 113, 114]: When $T \gg \nu$, the scalar field receives a contribution of $\hat{m}^2_{\rm th} = \lambda T^2/4!$ to its (initially negative) mass squared: $-\nu^2 \to -\nu^2 + \lambda T^2/4!$ or, equivalently,

$$V_{\rm cl} \to V_{\rm eff}(T) = V_{\rm cl} + \tfrac{1}{2}\hat{m}^2_{\rm th}\phi^2. \tag{3.21}$$

As a result, the minimum of $V_{\rm cl}$ becomes $\phi_{\rm min} = 0$ for $T \geq T_c = \sqrt{4!\nu^2/\lambda}$. Since

$$P = -V_{\rm eff}(T)\Big|_{\phi_{\rm min}} = \frac{\lambda}{384}\theta(T_c - T) \times (T_c^2 - T^2)^2, \tag{3.22}$$

the phase transition is of second order, i.e., there is no discontinuity in the first derivative of the pressure (in the entropy), but only in its second derivative, the specific heat.

The above *effective potential* can be derived more directly and systematically from the partition function evaluated at spatially constant field configurations $\bar{\phi}$,

$$V_{\rm eff}(\bar{\phi}; T) = -\frac{1}{\beta V}\ln Z\Big|_{\bar{\phi}}. \tag{3.23}$$

At one-loop order the temperature-dependent contribution of a scalar field with (field-dependent) mass $m^2(\bar{\phi}) = V''_{\rm cl}(\bar{\phi})$ is

$$\begin{aligned} V^{(1)}_{\rm eff}(\bar{\phi}; T) &= V_{\rm cl} + T\int \frac{d^3p}{(2\pi)^3}\ln\left(1 - e^{-\omega/T}\right) \quad \text{with } \omega = \sqrt{p^2 + m^2(\bar{\phi})} \\ &= V_{\rm cl} - \frac{\pi^2 T^4}{90} + \frac{1}{24}m^2(\bar{\phi})T^2 - \frac{1}{12\pi}m^3(\bar{\phi})T + \ldots \end{aligned} \tag{3.24}$$

The first field-dependent correction term reproduces the effective potential of (3.21). Subsequent terms in the high-T expansion as well as higher order loop contributions require a resummation of the thermal mass of the scalar field:

$$m^2(\bar{\phi}) \to m^2_{\rm eff}(\bar{\phi}; T) = m^2(\bar{\phi}) + \hat{m}^2_{\rm th}. \tag{3.25}$$

To avoid over-counting, this resummation has to take place either through the explicit introduction of thermal counterterms [114] or by using a self-consistent formalism like that of Φ-derivable approximation or the CJT effective action [115]. The latter allows to include superdaisy diagrams, which treated naively would give a spurious and in fact completely misleading contribution to the effective potential $\propto m(\bar{\phi})T^3$.

Insertion of (3.25) in (3.24) gives an effective potential which at the critical temperature has two degenerate minima, $\bar{\phi} = 0$ and $\bar{\phi} \sim \sqrt{\lambda}T_c$. However, at the second non-trivial minimum, $m_{\rm eff}(\bar{\phi}) \sim \lambda T$ and the loop expansion parameter of the perturbation series, which is $\lambda T/m_{\rm eff}$, ceases to be small. Hence, perturbation theory cannot decide whether there is a first-order phase transition as implied by a second degenerate minimum in the potential or not. In fact, universality arguments show that the phase transition must be second order for a ϕ^4 theory with Z_2 symmetry [61].

The situation is different in gauge theories. Again, resummed perturbation theory signals a first order phase transition, but when the Higgs mass is small compared to the massive vector-boson mass, it is sufficiently strongly first-order so that the perturbation series does not break down where the degenerate minima appear. The (resummed) effective potential has been calculated perturbatively to one-loop

[116, 117, 118] and two-loop order [114, 119, 120] (see also [121]). Of particular interest is the question whether the electroweak sector of the Standard Model (or extensions thereof) admits a first order phase transition, which would be of great interest for baryogenesis in the Early Universe [122]. Nonperturbative studies indeed confirmed a first-order transition for sufficiently small Higgs masses [123, 124], but found an endpoint [125, 126, 127] at Standard Model Higgs mass $m_H \lesssim 80$ GeV, so that a first-order phase transition in the Standard Model is excluded by the current experimental bounds on the Higgs mass (though not in possible (supersymmetric) extensions of the Standard Model [128, 129]).

4. QCD thermodynamics

The (resummed) perturbative evaluation of the thermodynamic potential of QCD at high temperature has been pushed in recent years up to the order $g^6 \ln g$ [130, 131, 132, 133, 134, 135, 136, 137]. At higher orders in g this is much facilitated by the possibility to employ effective field theory methods which in this case lead to a dimensional reduction to a 3-dimensional Yang-Mills plus adjoint Higgs theory [75]. Completion of these results at order g^6 is in fact impossible without inherently nonperturbative input, but further progress has been made most recently by the extension to nonzero quark chemical potential within dimensional reduction [138].

At zero temperature and high chemical potential the pressure is known to order g^4 [139, 140, 141, 142, 138], while the low-temperature expansion of the pressure leads to the phenomenon of non-Fermi-liquid behaviour of entropy and specific heat [143].

4.1. Dimensional reduction

Dimensional reduction in hot QCD leads to an effective three-dimensional Lagrangian [75, 76, 77]

$$\mathcal{L}_E = \tfrac{1}{2}\,\mathrm{tr}\, F_{ij}^2 + \mathrm{tr}[D_i, A_0]^2 + m_E^2\,\mathrm{tr}\, A_0^2 + \tfrac{1}{2}\lambda_E(\mathrm{tr}\, A_0^2)^2 + \ldots \tag{4.1}$$

where the parameters are determined perturbatively by matching [136, 144]. In lowest orderΣ and at zero chemical potential one has a dimensionful coupling $g_E^2 = g^2 T$ and [77]

$$m_E^2 = (1 + N_f/6) g^2 T^2, \qquad \lambda_E = \frac{9 - N_f}{12\pi^2} g^4 T, \tag{4.2}$$

though λ_E starts to contribute to the pressure only at order g^6. At this order, however, the self-interactions of the massless magnetostatic gluons start to contribute, and these contributions are inherently non-perturbative because the three-dimensional theory for the zero modes $A_i(\vec{x})$ is a confining theory [149, 150, 151].

The thermal pressure of the 4-dimensional theory can be decomposed into contributions from the hard modes $\sim T$, calculable by standard perturbation theory, and soft contributions governed by (4.1) which involves both perturbatively calculable contributions up to order $g^5 T^4$ and the nonperturbative ones starting at order $g^6 T^4$.

In reference [136] the effective theory based on (4.1) has been used to organize and reproduce the perturbative calculation of the thermal pressure to order g^5 of references [133, 134]. This turns out to be particularly elegant when dimensional

Σ Some higher-dimension terms in the effective theory (4.1) have been determined in references [78, 145, 146], and, including the electroweak sector, in references [147, 148].

regularization is used to provide both the UV and IR cutoffs of the original and effective field theories.

To order g^4, the contribution of the hard modes can then be written as [136]

$$P_{\rm hard} = \tfrac{8\pi^2}{45}T^4\Big\{\left(1+\tfrac{21}{32}N_f\right) - \tfrac{15}{4}\left(1+\tfrac{5}{12}N_f\right)\frac{\alpha_s}{\pi} + \Big\{244.9 + 17.24N_f - 0.415N_f^2 + 135\left(1+\tfrac{1}{6}N_f\right)\ln\frac{\bar\mu}{2\pi T} - \tfrac{165}{8}\left(1+\tfrac{5}{12}N_f\right)\left(1-\tfrac{2}{33}N_f\right)\ln\frac{\bar\mu}{2\pi T}\Big\}\left(\frac{\alpha_s}{\pi}\right)^2\Big\}. \tag{4.3}$$

In the first logarithm the dimensional regularization scale $\bar\mu$ is associated with regularization in the infrared and thus has to match a similar logarithm in the effective theory, whereas the second logarithm is from UV and involves the first coefficient of the beta function.

Indeed, calculating the pressure contribution of the soft sector described by (4.1) in dimensional regularization gives, to three-loop order (neglecting λ_E-contributions)

$$P_{\rm soft}/T = \frac{2}{3\pi}m_E^3 - \frac{3}{8\pi^2}\left(4\ln\frac{\bar\mu}{2m_E}+3\right)g_E^2 m_E^2 - \frac{9}{8\pi^3}\left(\frac{89}{24}-\frac{11}{6}\ln 2+\frac{1}{6}\pi^2\right)g_E^4\, m_E. \tag{4.4}$$

All the contributions to the pressure involving odd powers of g in (4.5) (as well as part of those involving even powers) are coming from the soft sector. Inserting the leading-order value (4.2) for m_E gives the QCD pressure up to and including order $g^4\ln g$; to obtain all the terms to order g^5, next-to-leading order corrections to the m_E-parameter have to be obtained by a matching calculation as given in reference [136]. The result is known in closed form [133, 134, 136], but we shall quote here only the case of SU(3) with N_f quark flavours and numerical values for the various coefficients:

$$\begin{aligned} P = \tfrac{8\pi^2}{45}T^4\Big\{&\left(1+\tfrac{21}{32}N_f\right) - \tfrac{15}{4}\left(1+\tfrac{5}{12}N_f\right)\frac{\alpha_s}{\pi} + 30\left[\left(1+\tfrac{1}{6}N_f\right)\left(\frac{\alpha_s}{\pi}\right)\right]^{3/2} \\ &+ \Big\{237.2 + 15.97N_f - 0.413N_f^2 + \tfrac{135}{2}\left(1+\tfrac{1}{6}N_f\right)\ln\left[\frac{\alpha_s}{\pi}(1+\tfrac{1}{6}N_f)\right] \\ &\quad - \tfrac{165}{8}\left(1+\tfrac{5}{12}N_f\right)\left(1-\tfrac{2}{33}N_f\right)\ln\frac{\bar\mu}{2\pi T}\Big\}\left(\frac{\alpha_s}{\pi}\right)^2 \\ &+ \left(1+\tfrac{1}{6}N_f\right)^{1/2}\Big[-799.2 - 21.96N_f - 1.926N_f^2 \\ &\quad + \tfrac{495}{2}\left(1+\tfrac{1}{6}N_f\right)\left(1-\tfrac{2}{33}N_f\right)\ln\frac{\bar\mu}{2\pi T}\Big]\left(\frac{\alpha_s}{\pi}\right)^{5/2} + \mathcal{O}(\alpha_s^3\ln\alpha_s)\Big\}. \end{aligned} \tag{4.5}$$

Here $\bar\mu$ is the renormalization scale parameter of the $\overline{\rm MS}$ scheme and $\alpha_s(\bar\mu)$ is the corresponding running coupling.

The coefficient of the $\alpha_s^3\ln\alpha_s$ term, the last in the pressure at high T and vanishing chemical potential that is calculable completely within perturbation theory, has recently been determined as [137, 152]

$$P\Big|_{g^6\ln g} = \tfrac{8\pi^2}{45}T^4\Big[1134.8 + 65.89N_f + 7.653N_f^2 - \tfrac{1485}{2}\left(1+\tfrac{1}{6}N_f\right)\left(1-\tfrac{2}{33}N_f\right)\ln\frac{\bar\mu}{2\pi T}\Big]\left(\frac{\alpha_s}{\pi}\right)^3\ln\frac{1}{\alpha_s}. \tag{4.6}$$

In order to obtain the $g^6 \ln g$-contribution (4.6) one also needs g_E^2 to order g^4 (given in reference [144]) and above all the four-loop contribution of the effective theory (4.1) which has recently been calculated analytically as [137]

$$P^{(4)}_{\rm soft}/T = N_g \frac{(N g_E^2)^3}{(4\pi)^4} \left[\left(\tfrac{43}{12} - \tfrac{157\pi^2}{768} \right) \ln \frac{\bar\mu}{g_E^2} + \left(\tfrac{43}{4} - \tfrac{491\pi^2}{768} \right) \ln \frac{\bar\mu}{m_E} + c \right] \tag{4.7}$$

up to a constant c that is strictly nonperturbative and needs to be determined by three-dimensional lattice calculations. Such calculations have been undertaken in reference [153], but they depend on an as yet undetermined 4-loop matching coefficient. At the moment the conclusion is that it is at least not excluded that the lattice results based on dimensional reduction can be matched to the full four-dimensional results at temperatures of a few times the transition temperature. For this reason, the most reliable results on the thermodynamics of hot QCD (particularly for pure-glue QCD) remain to date the four-dimensional lattice data. However, since inclusion of fermions is particularly easy in the dimensional reduction method, but computationally expensive in lattice gauge theory, a full three-dimensional prediction would clearly be most desirable.

4.2. Apparent convergence

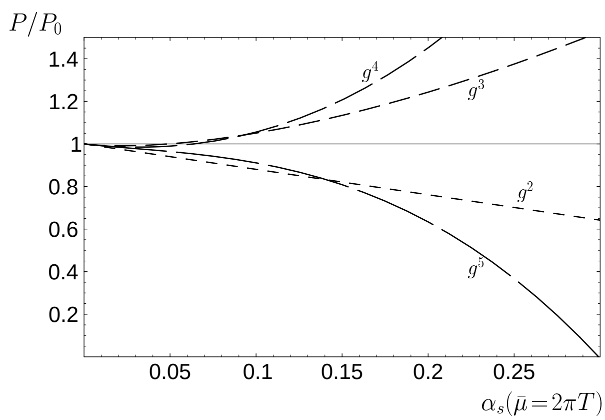


Figure 5. Strictly perturbative results for the thermal pressure of pure glue QCD normalized to the ideal-gas value as a function of $\alpha_s(\bar\mu = 2\pi T)$.

Figure 5 shows the outcome of evaluating the perturbative result (4.5) for the thermodynamic pressure at $N_f = 0$ to order α_s, $\alpha_s^{3/2}$, α_s^2, and $\alpha_s^{5/2}$, respectively, with a choice of $\bar\mu = 2\pi T$. Apparently, there is no convergence for $\alpha_s \gtrsim 0.05$ which in QCD corresponds to $T \lesssim 10^5 T_c$, where T_c is the deconfinement phase transition temperature. What is more, the numerical dependence on the renormalization scale $\bar\mu$ does not diminish as the order of the perturbative result is increased, but becomes more and more severe, as shown in figure 6, where $\alpha_s(\bar\mu)$ is determined by a 2-loop renormalization group equation and $\bar\mu$ is varied between πT and $4\pi T$. So there seems to be a complete loss of predictive power at any temperature of interest [133, 134, 136, 135].

To alleviate this situation, various mathematical extrapolation techniques have been tried, such as Padé approximants [85, 86, 156], self-similar approximants [157],

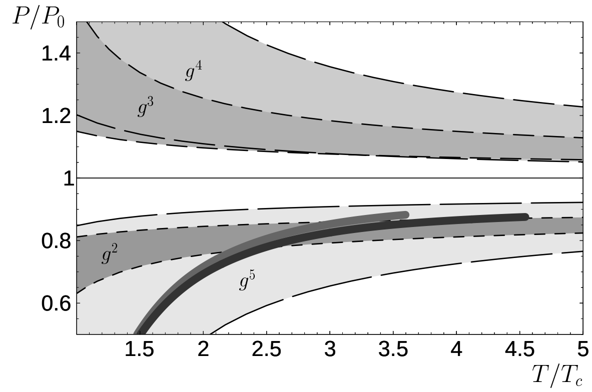


Figure 6. Strictly perturbative results for the thermal pressure of pure glue QCD as a function of T/T_c (assuming $T_c/\Lambda_{\overline{\rm MS}} = 1.14$). The various gray bands bounded by differently dashed lines show the perturbative results to order g^2, g^3, g^4, and g^5, using a 2-loop running coupling with $\overline{\rm MS}$ renormalization point $\bar{\mu}$ varied between πT and $4\pi T$. The thick dark-grey line shows the continuum-extrapolated lattice results from reference [154]; the lighter one behind that of a lattice calculation using an RG-improved action [155].

and Borel resummation [158, 159], however with limited success. While in the scalar toy model of Sect. 3.5, Padé approximants work remarkably well, in the QCD case there is a problem how to handle logarithms of the coupling and, perhaps related to that, the numerical results obtained so far appear less satisfactory.

When compared with the perturbative results, it is however remarkable that the next-to-leading result to order g^2 performs rather well at temperatures $\gtrsim 2T_c$, though the higher-order results prove that perturbation theory is inconclusive. Moreover, simple quasiparticle models which describe the effective gluonic degrees of freedom by $2N_g$ ($N_g = N^2 - 1$) scalar degrees of freedom with asymptotic masses taken from a HTL approximation can be used quite successfully to model the lattice data by fitting the running coupling [160, 161, 162, 163].

In fact, we have seen in section 3.5 that already in the simplest scalar model resummed perturbation theory gives rather poorly convergent results, and that fairly simple reorganizations as in (3.13) lead to dramatic improvements of the situation.

As has been noted in references [137, 164], in the calculation as organized through the dimensionally reduced effective theory (4.1), the large scale dependence of strict perturbation theory can be significantly reduced when the perturbative values of the effective parameters are kept as they appear in (4.4), without expanding and truncating the final result to the considered order in the coupling. What is more, the unexpanded two-loop and three-loop contributions from the soft sector lead to results which no longer exceed the ideal-gas result (as the strictly perturbative results to order g^3 and g^4 do), and their (sizable) scale dependence diminishes by going from two to three-loop order [164].

At three-loop order, it is in fact possible to eliminate the scale dependence altogether by a principle of minimal sensitivity. The result in fact agrees remarkably well with the 4-d lattice results down to $\sim 2.5T_c$ as shown in figure 7.

The four-loop order result depends on the unknown constant c in (4.7). However,

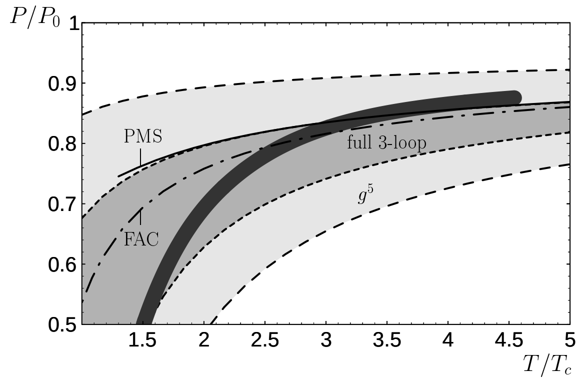


Figure 7. Three-loop pressure in pure-glue QCD with unexpanded effective-field-theory parameters when $\bar{\mu}$ is varied between πT and $4\pi T$ (medium-gray band); the dotted lines indicate the position of this band when only the leading-order result for m_E is used. The broad light-gray band underneath is the strictly perturbative result to order g^5 with the same scale variations. The full line gives the result upon extremalization (PMS) with respect to $\bar{\mu}$ (which does not have solutions below $\sim 1.3T_c$); the dash-dotted line corresponds to fastest apparent convergence (FAC) in m_E^2, which sets $\bar{\mu} \approx 1.79\pi T$. (Taken from [164])

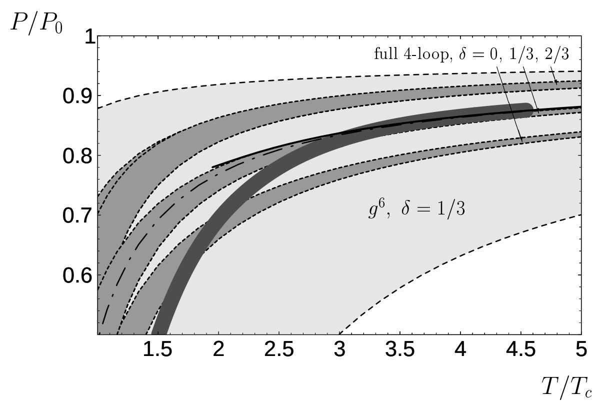


Figure 8. Like figure 7, but extended to four-loop order by including the recently determined $g^6 \ln(1/g)$ contribution of [137] together with three values for the undetermined constant δ in $[g^6 \ln(1/g)+\delta]$. The broad light-gray band underneath is the strictly perturbative result to order g^6 corresponding to the central value $\delta = 1/3$, which has a larger scale dependence than the order g^5 result in figure 7; the untruncated results on the other hand show rather small scale dependence. The full line gives the untruncated result with $\delta = 1/3$ and $\bar{\mu}$ fixed by PMS (which does not have solutions below $\sim 1.9T_c$); the dash-dotted line corresponds to fastest apparent convergence (FAC) in m_E^2, which sets again to $\bar{\mu} \approx 1.79\pi T$. (Taken from [164])

it is at least not excluded that c could be such that the four-loop result is also close to the 4-d lattice results [137, 165]. In fact, while a strictly perturbative treatment leads to increased scale dependence compared to 3-loop order, keeping the soft contributions unexpanded in g further diminishes the scale dependence [164] as shown in figure 8.

While this goes only minimally beyond a strictly perturbative treatment, it strongly suggests that perturbative QCD at high temperature, when supplemented by appropriate resummation of soft physics, is *not* limited to $T \gg 10^5 T_c$ as previously thought [133, 135], but seems capable of quantitative predictions at temperatures of possibly only a few times the transition temperature.

4.3. Finite chemical potential

The calculation of the thermodynamical potential using dimensional reduction can in fact be extended to nonvanishing quark chemical potential μ_q, provided however that $T \gg m_E$. This has been carried out up to and including order $g^6 \ln g$ in reference [138], and a discussion of the issue of apparent convergence can be found in reference [166].

A non-zero chemical potential modifies the parameters of the effective theory, in particular the mass parameter m_E, which is altered already at leading order according to (5.18) below. In addition there are new, C-odd terms in the effective Lagrangian. The one with smallest dimension in nonabelian theories reads [167, 168, 169]

$$\mathcal{L}_E^{(\mu)} = i \frac{g^3}{3\pi^2} \sum_q \mu_q \,\mathrm{tr}\, A_0^3. \tag{4.8}$$

In Abelian theories there is also a linear term involving $\mathrm{tr}\, A_0$ which has been discussed e.g. in [170].

In general the effects of these additional C-odd terms are small compared to the C-even operators in (4.1) which depend on the chemical potential through its parameters. One quantity which is determined to leading order by the operator (4.8) is the flavour off-diagonal quark number susceptibility at zero chemical potential [171]

$$\chi_{ij} \equiv \frac{\partial^2 P}{\partial \mu_i \partial \mu_j}. \tag{4.9}$$

When quark masses are negligible, all off-diagonal components are equal at $\mu_i = 0$. Denoting them by $\tilde{\chi}$, the leading-order term involves a logarithmic term coming from the exchange of three electrostatic gluons and is given by [171]

$$\tilde{\chi} \simeq -\frac{(N^2-1)(N^2-4)}{384N} \left(\frac{g}{\pi}\right)^6 T^2 \ln \frac{1}{g}. \tag{4.10}$$

where N is the number of colours. This vanishes in SU(2) gauge theory, but not in QED, where (in the ultrarelativistic limit) [171]

$$\tilde{\chi}\Big|_{\mathrm{QED}} \simeq -\frac{e^6}{24\pi^6} T^2 \ln \frac{1}{e}. \tag{4.11}$$

On the other hand, the diagonal quark susceptibilities have a perturbative expansion whose first few terms are given by

$$\frac{\chi}{\chi_0} = 1 - \frac{1}{2}\frac{3}{N}\frac{N_g}{8}\left(\frac{g}{\pi}\right)^2 + \frac{3}{N}\frac{N_g}{8}\sqrt{\frac{N}{3}+\frac{N_f}{6}}\left(\frac{g}{\pi}\right)^3 + \frac{3}{4}\frac{N_g}{8}\left(\frac{g}{\pi}\right)^4 \log\frac{1}{g} + \mathcal{O}(g^4), \tag{4.12}$$

with $\chi_0 = NT^2/3$ the ideal gas value and $N_g = N^2 - 1$. The higher-order coefficients have been calculated by Vuorinen [172] up to and including order $g^6 \ln(1/g)$. The problem with apparent convergence is similar if somewhat less severe than in the case of the pressure at zero chemical potential discussed above.

In contrast to the pressure, however, the coefficient of the order g^6 term in χ is not sensitive to nonperturbative chromomagnetostatic physics and thus

calculable in perturbation theory, though not yet available. Its determination would in fact be of some interest in view of the important progress that has recently been made with the inclusion of small chemical potentials in lattice gauge theory [173, 174, 175, 176, 177, 178, 179, 180].

4.4. Low temperature and high chemical potential

At large chemical potential but $T \lesssim g\mu_q$, dimensional reduction does not occur. In this case a fully four-dimensional computation has to be performed.

The perturbative result up to and including order g^4 at zero temperature has been calculated by Freedman and McLerran [139, 140, 141] and by Baluni [142] more than a quarter of a century ago.

This result was originally given in a particular gauge-dependent momentum subtraction scheme. In order to convert it to the gauge-independent $\overline{\rm MS}$ scheme, one needs to replace the scale parameter μ_0 in reference [141] (M in references [142, 8]) according to [99, 181]

$$\mu_0 = \bar\mu \exp\{[(151)N - 40N_f]/[24(11N - 2N_f)]\}\,. \tag{4.13}$$

Furthermore, the order g^4 contributions involved two integrals that were evaluated only numerically with sizable error bars. This calculation was recently repeated in reference [138], with one of these integrals evaluated analytically and the other with very high accuracy.

Specialized to $N = 3$ and uniform chemical potential for N_f quark flavours, the result for the pressure at zero temperature to order g^4 in the $\overline{\rm MS}$ scheme reads

$$P = \frac{N_f\mu_q^4}{4\pi^2}\Big\{1 - 2\frac{\alpha_s(\bar\mu)}{\pi} - \Big[18 - 11\log 2 - 0.53583N_f + N_f \log\frac{N_f\alpha_s(\bar\mu)}{\pi} \\ +\ (11 - \frac{2}{3}N_f)\log\frac{\bar\mu}{\mu_q}\ \Big](\frac{\alpha_s(\bar\mu)}{\pi})^2 + O(\alpha_s^3\log\alpha_s)\Big\}, \tag{4.14}$$

where μ_q is the (common) quark chemical potential, not to be confused with the renormalization scale $\bar\mu$ of the $\overline{\rm MS}$ scheme.

At zero temperature and large chemical potential, perturbation theory is not hampered by a perturbative barrier at order g^6 so that higher-order corrections are in principle calculable, but not yet available (except for one term $\propto N_f^3 g^6$ extracted numerically from the solvable large-N_f limit of QCD [71]).

However, at small but finite temperature $T \lesssim g\mu$, the only weakly screened low-frequency transverse gauge-boson interactions (see (5.19) below) lead to a qualitative deviation from the Fermi liquid behaviour of relativistic systems described in reference [182]. In particular, the low-temperature limit of entropy and specific heat does not vanish linearly with temperature, but there is a positive contribution proportional to $\alpha T \ln T^{-1}$ [183, 184, 185], which implies that at sufficiently small temperature the entropy exceeds the ideal-gas result. In reference [143] this effect was most recently calculated beyond the coefficient of the leading log obtained in [183, 185]‖. The complete result for the entropy below order $T^3 \ln T$, where regular Fermi-liquid corrections enter, is given by (for SU(3) and with numerically evaluated coefficients

‖ The coefficient of the $\alpha T \ln T^{-1}$ term given in the original paper [183] was found to be lacking of a factor of 4.

which are known in closed form [143])

$$\mathcal{S} = \mu_q^2 T \left\{ \frac{NN_f}{3} + \frac{\alpha_s N_f N_g}{18\pi} \ln\left(2.2268\sqrt{\frac{\alpha_s N_f}{\pi}}\frac{\mu_q}{T}\right) - 0.17286 N_g \left(\frac{\alpha_s N_f}{\pi}\right)^{\frac{2}{3}} \left(\frac{T}{\mu_q}\right)^{\frac{2}{3}} \right.$$
$$\left. + 0.13014 N_g \left(\frac{\alpha_s N_f}{\pi}\right)^{\frac{1}{3}} \left(\frac{T}{\mu_q}\right)^{\frac{4}{3}} \right\} + O(T^3 \ln T), \quad T \ll g\mu_q, \tag{4.15}$$

and turns out to involve also fractional powers of the temperature as well as the coupling. (The corresponding result in QED is obtained by replacing $N_g \to 1$ and $\alpha_s N_f \to 2\alpha$.) Equation (4.15) is the beginning of a perturbative expansion provided $T/\mu_q \ll g$, e.g. $T/\mu_q \sim g^{1+\delta}$ with $\delta > 0$. The corresponding contribution to the pressure is then of the order $g^{4+2\delta} \ln g$ and thus of higher order than the terms evaluated in (4.14). But in the entropy the zero-temperature limit of the pressure drops out and the nonanalytic terms found in (4.15) become the leading interaction effects when $T/\mu_q \ll g$. For exponentially small $T/(g\mu_q) \sim \exp(-\#/g^2)$, they eventually become comparable to the ideal-gas part and the perturbative treatment breaks down (in QCD one in fact expects a breakdown of perturbation theory already at the order of $\exp(-\#/g)$ for colour superconducting quarks, cf. section 5.3).

Originally, the anomaly in the specific heat was discussed for a nonrelativistic electron gas with the expectation that this effect may be too small for experimental detection [183]. In QCD, however, it is numerically much more important, not only because $\alpha_s \gg \alpha_{\rm QED}$, but also because of the relatively large factor $N_g = 8$. Consequently, it may play some role in the thermodynamics of (proto-)neutron stars, if those have a normal (non-superconducting) quark matter component.

For potential phenomenological applications in astrophysical systems, the specific heat C_v at constant volume and number density is of more direct interest, which however differs from the logarithmic derivative of the entropy only by subleading terms [186]:

$$\mathcal{C}_v \equiv C_v/V = T \left\{ \left(\frac{\partial \mathcal{S}}{\partial T}\right)_{\mu_q} - \left(\frac{\partial \mathcal{N}}{\partial T}\right)_{\mu_q}^2 \left(\frac{\partial \mathcal{N}}{\partial \mu_q}\right)_T^{-1} \right\} = T \left(\frac{\partial \mathcal{S}}{\partial T}\right)_{\mu_q} + O(T^3)\,, \tag{4.16}$$

where $\mathcal{N}$ is the number density. In QCD this gives

$$\mathcal{C}_v = \mu_q^2 T \left\{ \frac{NN_f}{3} + \frac{\alpha_s N_f N_g}{18\pi} \ln\left(1.9574\sqrt{\frac{\alpha_s N_f}{\pi}}\frac{\mu_q}{T}\right) - 0.288095 N_g \left(\frac{\alpha_s N_f}{\pi}\right)^{\frac{2}{3}} \left(\frac{T}{\mu_q}\right)^{\frac{2}{3}} \right.$$
$$\left. + 0.3036697 N_g \left(\frac{\alpha_s N_f}{\pi}\right)^{\frac{1}{3}} \left(\frac{T}{\mu_q}\right)^{\frac{4}{3}} \right\} + O(T^3 \ln T), \quad T \ll g\mu_q. \tag{4.17}$$

The results (4.15) and (4.17) imply an excess over their respective ideal-gas values. They depend on having $T \ll g\mu_q$, and in this regime the low-temperature series has a small expansion parameter $T/(g\mu_q)$; for $T \gg g\mu_q$ the standard exchange term gives the leading-order interaction contribution [8]

$$\mathcal{C}_v \simeq \mathcal{S} = \mu_q^2 T \left\{ \frac{NN_f}{3} - N_g N_f \frac{\alpha_s}{4\pi} + O(\alpha_s^4) \right\} + O(T^3), \tag{4.18}$$

so that for larger temperature there is a reduction compared to the ideal-gas result. A nonmonotonic behaviour of the entropy as a function of T which interpolates between (4.15) and (4.18) has indeed been found in the numerical evaluation of the exactly solvable large-N_f limit of QED and QCD [70], and there the domain where

the entropy has the anomalous feature of exceeding the ideal-gas value is given by $T/\mu_q \lesssim g\sqrt{N_f}/30$.

Non-Fermi-liquid corrections to $\mathcal{C}_v$ in the context of ultrarelativistic QED and QCD have also been considered in reference [187], however the result obtained therein does not agree with (4.17). When expanding perturbatively the renormalization-group resummed result of reference [187], it would imply a leading nonanalytic $\alpha T^3 \ln T$ term, which is in fact the kind of nonanalytic terms that appear also in regular Fermi-liquids [188]. However, reference [187] did not evaluate all contributions $\propto \alpha T$, which were considered to be free of nonanalytic terms.

5. The quasi-particle spectrum in gauge theories

With the exception of static quantities in the high-temperature limit, where dimensional reduction is applicable, a systematic calculation of observables in thermal field theory, such as reaction rates, transport coefficients, and even the thermodynamic potential at low temperatures and high chemical potential require the determination and consistent inclusion of medium effects on the dynamical propagators in the theory.

As we have seen in the example of scalar field theory in Sect. 3.2, the interactions with the particles of the heat bath modify the spectrum of elementary excitations. The poles of the propagator, which determines the linear response of the system under small disturbances, receive thermal corrections which, in general, introduce a mass gap for propagating modes and screening for non-propagating ones, even when the underlying field theory is massless. The simple $\mathrm{O}(N \to \infty)$ ϕ^4 model considered in Sect. 3.2 is in fact somewhat misleading in that there the self-energy is a real quantity to all orders in the coupling, whereas in a nontrivial quantum field theory a nonvanishing width for propagating modes is unavoidable [189]. Nonetheless, the typical situation in perturbation theory is that the width is parametrically smaller than the thermal mass at a given momentum, and that the former can be treated to some extent perturbatively. It should be noted, however, that such quasiparticle excitations need not correspond to simple poles on the unphysical sheet. There could instead be branch points, branch singularities, essential singularities, or even no singularities at all [190].

Moreover, in the case of gauge theories it is a priori not clear whether the thermal corrections to the various propagators encode physical information or not. In Abelian gauge theory, the photon propagator is linearly related to correlators of the gauge-invariant electromagnetic field strength and so has a direct physical interpretation; matter fields on the other hand already transform non-trivially under gauge transformation and indeed their propagator is a gauge-fixing dependent quantity. In non-Abelian gauge theories, the gauge bosons carry colour charge and their propagator is also gauge dependent. Correlators of the non-Abelian field strength are not gauge independent either.

Still, even in gauge-dependent quantities there may be gauge-independent information. Indeed, in [191, 192] (see also [193] for a more detailed recent review) it has been shown that the singularity structure (location of poles and branch singularities) of certain components of gauge and matter propagators are gauge independent when all contributions to a given order of a systematic expansion scheme are taken into account.

Another example for gauge-independent content in gauge-dependent quantities is provided by the high-temperature limit of self-energies and more-point correlation

functions with small external momenta. Those are related to forward-scattering amplitudes [194, 195] of on-shell plasma constituents and are therefore completely gauge independent. They can form the building blocks of an effective theory at soft scales (with respect to the temperature), as we shall describe further below. Before doing so, we review the structure of the various propagators in a gauge theory.

5.1. Gauge-boson propagator

The self-energy of gauge bosons is a symmetric tensor, the so-called polarization tensor, which is defined by

$$-\Pi^{\mu\nu} = G^{-1\,\mu\nu} - G_0^{-1\,\mu\nu}. \tag{5.1}$$

(In the real-time formalism, where this should be defined first as a 2×2 matrix relation, we assume that Π has been extracted after diagonalization, for example in the $F, \bar{F}$ basis according to (2.27).)

If the gauge fixing procedure does not break rotational invariance in the plasma rest frame, this Lorentz tensor can be decomposed in terms of four independent tensors and associated structure functions,

$$-\Pi^{\mu\nu} = \Pi_A A^{\mu\nu} + \Pi_B B^{\mu\nu} + \Pi_C C^{\mu\nu} + \Pi_D D^{\mu\nu}, \tag{5.2}$$

with, in momentum space, $A^{\mu\nu}$ being the spatially transverse tensor (2.33), and the others chosen as [131, 151, 196, 197, 11, 55]

$$B^{\mu\nu}(k) = \frac{\tilde{n}^\mu \tilde{n}^\nu}{\tilde{n}^2} \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} - A^{\mu\nu}(k), \tag{5.3}$$

$$C^{\mu\nu}(k) = \frac{1}{|\mathbf{k}|} \left\{ \tilde{n}^\mu k^\nu + k^\mu \tilde{n}^\nu \right\}, \tag{5.4}$$

$$D^{\mu\nu}(k) = \frac{k^\mu k^\nu}{k^2}, \tag{5.5}$$

where $\tilde{n}^\mu = (\eta^{\mu\sigma} - k^\mu k^\sigma / k^2)\delta^0_\sigma$ in the plasma rest frame.

Only A and B are transverse with respect to the four-momentum k, $A^{\mu\nu}k_\nu = B^{\mu\nu}k_\nu = 0$. C obeys the weaker relation $C^{\mu\nu}k_\mu k_\nu = 0$, and D projects onto k. This particular basis is a convenient choice because A, B, and D are idempotent and mutually orthogonal; C is only orthogonal to A, but its product with the other tensors has vanishing trace.

In Abelian gauge theory with linear gauge fixing, the Ward identities imply transversality of the polarization tensor, $\Pi^{\mu\nu}k_\nu \equiv 0$. In non-Abelian gauge theories transversality holds only in certain gauges such as axial gauges [198] (with only the temporal axial gauge respecting rotational invariance) and background-covariant gauges [199, 200, 201]. Contrary to the experience at zero temperature, at finite temperature the polarization tensor turns out to be non-transverse already at one-loop order in general covariant and Coulomb gauges [197, 202], with the fortuitous exception of Feynman gauge (at one-loop order).

For a gauge breaking Lagrangian (2.34) BRS invariance only requires that [203, 11, 204, 205]

$$\tilde{f}_\mu \tilde{f}_\nu G^{\mu\nu} = -\alpha\,. \tag{5.6}$$

This entails that [204, 205]

$$\Pi_D \left(k^2 - \Pi_B \right) = \Pi_C^2 \tag{5.7}$$

so that generally the polarization tensor contains 3 independent structure functions. Equation (5.7) also implies that at one-loop order $\Pi_D \equiv 0$, but not beyond in those gauges where $\Pi_C \not\equiv 0$.

Rotationally invariant gauge fixing vectors can be written generally as $\tilde{f}^\mu = \tilde{\beta}(k)k^\mu + \tilde{\gamma}(k)\tilde{n}^\mu$ with $\tilde{\beta} \neq 0$. This includes covariant gauges ($\tilde{\beta} = 1, \tilde{\gamma} = 0$), Coulomb gauges ($\tilde{\beta} = \tilde{n}^2, \tilde{\gamma} = -k^0$), or temporal gauges ($\tilde{\beta} = k^0/k^2, \tilde{\gamma} = 1$).

For these, the structure functions in the full propagator

$$-G^{\mu\nu} = \Delta_A A^{\mu\nu} + \Delta_B B^{\mu\nu} + \Delta_C C^{\mu\nu} + \Delta_D D^{\mu\nu} \tag{5.8}$$

are determined by

$$\Delta_A = [k^2 - \Pi_A]^{-1} \tag{5.9}$$

$$\Delta_B = [k^2 - \Pi_B - \frac{2\tilde{\beta}\tilde{\gamma}|\mathbf{k}|\Pi_C - \alpha\Pi_C^2 + \tilde{\gamma}^2\tilde{n}^2\Pi_D}{\tilde{\beta}^2 k^2 - \alpha\Pi_D}]^{-1} \tag{5.10}$$

$$\Delta_C = -\frac{\tilde{\beta}\tilde{\gamma}|\mathbf{k}| - \alpha\Pi_C}{\tilde{\beta}^2 k^2 - \alpha\Pi_D}\Delta_B \tag{5.11}$$

$$\Delta_D = \frac{\tilde{\gamma}^2\tilde{n}^2 + \alpha(k^2 - \Pi_B)}{\tilde{\beta}^2 k^2 - \alpha\Pi_D}\Delta_B \tag{5.12}$$

5.1.1. Gauge independence of singularities In [191, 192] it has been shown that under variations of the gauge fixing parameters (in our case $\alpha, \tilde{\beta}, \tilde{\gamma}$) one has "gauge dependence identities" (generalized Nielsen identities [206, 207, 208, 209]) which are of the form

$$\delta\Delta_A^{-1}(k) = \Delta_A^{-1}\left[-A^\mu_\nu(k)\delta X^\nu_{,\mu}(k)\right] \equiv \Delta_A^{-1}(k)\delta Y(k), \tag{5.13}$$

$$\delta\Delta_B^{-1}(k) = \Delta_B^{-1}\left[-\frac{\tilde{n}^\mu}{\tilde{n}^2} + \frac{\tilde{\gamma}\tilde{\beta} - \alpha\Pi_C/|\vec{k}|}{\tilde{\beta}^2 k^2 - \alpha\Pi_D}k^\mu\right] 2\tilde{n}_\nu \delta X^\nu_{,\mu} \equiv \Delta_B^{-1}(k)\delta Z(k), \tag{5.14}$$

where $\delta X^\nu_{,\mu}$ has a diagrammatic expansion which is one-particle-irreducible except for at most one Faddeev-Popov ghost line. No such relation exists for Δ_C or Δ_D.

Now if δY and δZ are regular on the two "mass-shells" defined by $\Delta_A^{-1} = 0$ and $\Delta_B^{-1} = 0$, the relations (5.13,5.14) imply that the locations of these particular singularities of the gluon propagator are gauge fixing independent, for if $\Delta_A^{-1} = 0 = \Delta_B^{-1}$ then also $\Delta_A^{-1} + \delta\Delta_A^{-1} = 0 = \Delta_B^{-1} + \delta\Delta_B^{-1}$.

In the case of Δ_B, singularities in $\delta Z(k)$ include a kinematical pole $1/k^2$ hidden in the $\tilde{n}$'s and the manifestly gauge-dependent Δ_D (cf. (5.12)). Excluding these obvious gauge artefacts, everything depends on whether the possible singularities of $\delta X^\nu_{,\mu}$ could coincide with the expectedly physical dispersion laws $\Delta_A^{-1} = 0$ and $\Delta_B^{-1} = 0$. Because $\delta X^\nu_{,\mu}$ is one-particle reducible with respect to Faddeev-Popov ghosts, the singularities of the latter have to be excluded, too. However, these are generically different from those that define the spatially transverse and longitudinal gauge-boson quasi-particles. Indeed, in leading-order thermal perturbation theory the Faddeev-Popov ghost self energy and the physical self energies receive contributions carrying different powers of temperature or chemical potential and therefore have independent and generically different dispersion laws.

However, $\delta X^\nu_{,\mu}$ may develop singularities also from one-particle-irreducible subdiagrams, namely when one line of such a diagram is of the same type as the external one and the remaining ones are massless. This can give rise to infrared

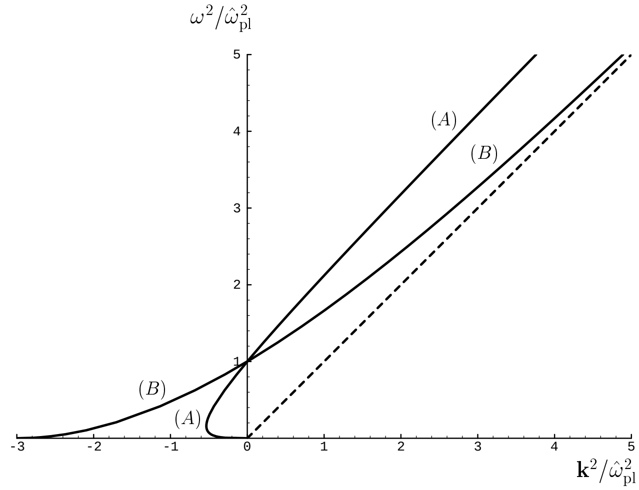


Figure 9. Location of poles in Δ_A and Δ_B of the hard-thermal-loop gauge propagator. The right part with $\mathbf{k}^2 \geq 0$ corresponds to propagating normal modes, the left part to (dynamical) screening.

or mass-shell singularities and seemingly constitute an obstruction to the gauge-independence proof [210, 211]. But such singularities will be absent as soon as an overall infrared cut-off is introduced, for example by restricting everything to a finite volume first. In every finite volume, this obstruction to the gauge-independence proof is then avoided, and $\Delta_A^{-1} = 0$ and $\Delta_B^{-1} = 0$ define gauge-independent dispersion laws if the infinite-volume limit is taken last of all [212].

5.1.2. Gauge-field quasi-particles at leading order It turns out that the leading-order contributions to $\Pi^{\mu\nu}$ for small frequencies and momenta $k^0, |\mathbf{k}| \ll T$ are entirely transverse ($\Pi_C = \Pi_D = 0$) and gauge-parameter independent. They are generated by one-loop diagrams with hard loop momentum and for this reason they are termed “hard-thermal-loops” (HTL) [213]. Their form is the same in Abelian [214, 215] and non-Abelian gauge theories [216, 217], and also in the presence of chemical potentials μ_f for fermions such that $\max(T, \mu_f) \gg k^0, |\mathbf{k}|$. This universal result reads

$$\hat{\Pi}_A = \tfrac{1}{2}(\hat{\Pi}_\mu{}^\mu - \hat{\Pi}_B) \tag{5.15}$$

$$\hat{\Pi}_B = -\frac{k^2}{\mathbf{k}^2}\hat{\Pi}_{00} \tag{5.16}$$

with

$$\hat{\Pi}_\mu{}^\mu = \hat{m}_D^2, \quad \hat{\Pi}_{00} = \hat{m}_D^2 \left(1 - \frac{k^0}{2|\mathbf{k}|} \ln \frac{k^0 + |\mathbf{k}|}{k^0 - |\mathbf{k}|}\right), \tag{5.17}$$

$$\hat{m}_D^2 = \left\{ \begin{array}{ll} e^2T^2/3 + e^2\mu_e^2/\pi^2 & \text{for QED}, \\ g^2NT^2/3 + \sum_f g^2\mu_f^2/(2\pi^2) & \text{for SU}(N) \text{ with } N_f \text{ flavours}. \end{array} \right. \tag{5.18}$$

The spectral representation of the resulting propagators $\Delta_{A,B}$ involves simple poles and continuous parts for $k^2 < 0$, which are given in detail in appendix A.1.

In figure 9 the location of the poles of the corresponding propagators Δ_A and Δ_B are displayed in quadratic scales. The light-cone is marked by a dashed line. A

simple mass hyperboloid would be given by a line parallel to the latter. Evidently, the dispersion laws of the HTL quasiparticle excitations are more complicated—they involve momentum-dependent masses: in the long-wavelength limit $|\mathbf{k}| \to 0$ there is a common lowest (plasma) frequency $\hat{\omega}_{\rm pl.} = \hat{m}_D/\sqrt{3}$ for propagating normal modes.

For larger frequencies and momenta $\omega, |\mathbf{k}| \gg \hat{m}_D$ it turns out that mode A approaches asymptotically a mass hyperboloid with mass $m_\infty = \hat{m}_D/\sqrt{2}$. So in this momentum region the physical spatially transverse polarizations of the gauge bosons acquire indeed a constant thermal mass. The additional mode B, whose dispersion curve approaches the light-cone exponentially, is found to have a spectral strength (the residue of the corresponding pole) that decays exponentially as $k/\hat{m}_D \to \infty$ [218], showing its exclusively collective nature.

For real $|\mathbf{k}|$ but $\omega^2 < \mathbf{k}^2$, $\hat{\Pi}_{\mu\nu}$ has an imaginary part $\sim \hat{m}_D^2$ from the logarithm in (5.17) which prevents the appearance of poles in this region. This imaginary part corresponds to the possibility of Landau damping, which is the transfer of energy from soft fields to hard plasma constituents moving in phase with the field [219, 6] and is an important part of the spectral density of HTL propagators. At higher, subleading orders of perturbation theory, it is, however, not protected against gauge dependences in nonabelian gauge theories, in contrast to the location of the singularities which determine the dispersion laws of quasi-particles. However, at asymptotically large times, Landau damping is (generically) dominated by the gauge-independent location of the branch cuts at $\omega = \pm|\mathbf{k}|$, resulting in power-law relaxation of perturbations [220]. There is also an exponential component of Landau damping due to a pole at purely imaginary ω and real $|\mathbf{k}|$, on the unphysical sheet reached by continuation through the branch cut between $\omega = \pm|\mathbf{k}|$ [221].

For real $\omega < \hat{\omega}_{\rm pl.}$, there are no poles for real $|\mathbf{k}|$, but instead for imaginary $\sqrt{\mathbf{k}^2} = i\kappa$, corresponding to exponential (dynamical) screening of (time-dependent) external sources. These poles, displayed on the left part of figure 9, are in fact closely related to the just mentioned poles at purely imaginary ω and real $|\mathbf{k}|$.

In the static limit $\omega \to 0$, only mode B is screened with (Debye) screening length $\hat{m}_D^{-1}$. This corresponds to exponential screening of (chromo-) electrostatic fields. The transverse mode A on the other hand is only weakly screened when $\omega \ll m_D$ with a frequency-dependent inverse screening length [217]

$$\kappa_A \simeq \left(\frac{\pi m_D^2 \omega}{4} \right)^{1/3}, \qquad \omega \ll m_D. \tag{5.19}$$

When $\omega \to 0$, mode A describes magnetostatic fields which are found to be completely unscreened in the HTL approximation. In QED the absence of magnetic screening is intuitively clear and can be proved rigorously to all orders of perturbation theory [215, 222], but not in the nonabelian case.

In QCD, the absence of a magnetostatic screening mass causes problems for perturbation theory as will be discussed further in Sect. 7.4. In fact, lattice simulations of gauge fixed propagators in nonabelian theories do find a screening behaviour in the transverse sector, though the corresponding singularity is evidently quite different from a simple pole [223].

5.2. Fermions

The fermion self-energy at non-zero temperature or density has one more structure function than usually. In the ultrarelativistic limit where masses can be neglected, it

can be parametrized by

$$\Sigma(\omega, \mathbf{k}) \;=\; a(\omega, |\mathbf{k}|)\,\gamma^0 \;+\; b(\omega, |\mathbf{k}|)\hat{\mathbf{k}} \cdot \boldsymbol{\gamma}. \tag{5.20}$$

(For a massive fermion, this would also include a mass correction, i.e., $\Sigma = a\gamma^0 + b\,\hat{\mathbf{k}} \cdot \boldsymbol{\gamma} + c\mathbf{1}$.) This can be rewritten as:

$$\gamma_0 \Sigma(\omega, \mathbf{k}) \;=\; \Sigma_+(\omega, |\mathbf{k}|)\,\Lambda_+(\hat{\mathbf{k}}) \;-\; \Sigma_-(\omega, |\mathbf{k}|)\,\Lambda_-(\hat{\mathbf{k}}), \tag{5.21}$$

where $\Sigma_\pm \equiv b \pm a$, and the spin matrices

$$\Lambda_\pm(\hat{\mathbf{k}}) \equiv \frac{1 \pm \gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}}}{2} \tag{5.22}$$

project onto spinors whose chirality is equal (Λ_+), or opposite (Λ_-), to their helicity. Dyson's equation $S^{-1} = -\not{k} + \Sigma$ then implies

$$\gamma_0 S^{-1} = \Delta_+^{-1} \Lambda_+ \;+\; \Delta_-^{-1} \Lambda_-, \tag{5.23}$$

with $\Delta_\pm^{-1} \equiv -[\omega \mp (|\mathbf{k}| + \Sigma_\pm)]$. This is trivially inverted to yield the fermion propagator

$$S\gamma_0 = \Delta_+ \Lambda_+ + \Delta_- \Lambda_-. \tag{5.24}$$

whose singularities are conveniently summarized by the equation

$$\det S^{-1}(\omega, \mathbf{k}) = 0, \tag{5.25}$$

where the determinant refers to spinor indices. The potential gauge dependences of the singularities of S are described by an identity of the form [192]

$$\delta \det S^{-1}(\omega, \mathbf{k}) = \det S^{-1}(\omega, \mathbf{k})\; \delta\, \mathrm{tr}\, X(\omega, \mathbf{k}), \tag{5.26}$$

where $\delta\, \mathrm{tr}\, X(\omega, \mathbf{k})$ again has a diagrammatic expansion which is one-particle-irreducible except for at most one Faddeev-Popov ghost line. The same reasoning as in Sect. 5.1.1 (with similar qualifications) leads to the conclusion that the positions of the singularities of the fermion propagator are gauge-fixing independent [192].

In the HTL approximation, the fermion self-energies are once again gauge-independent in their entirety. Explicitly, they read [224, 225, 226, 227, 228, 229]:

$$\hat{\Sigma}_\pm(\omega, |\mathbf{k}|) \;=\; \frac{\hat{M}^2}{k} \left(1 \;-\; \frac{\omega \mp |\mathbf{k}|}{2|\mathbf{k}|} \;\log\, \frac{\omega + |\mathbf{k}|}{\omega - |\mathbf{k}|} \right), \tag{5.27}$$

where $\hat{M}^2$ is the plasma frequency for fermions, i.e., the frequency of long-wavelength ($k \to 0$) fermionic excitations:

$$\hat{M}^2 = \frac{g^2 C_f}{8} \left(T^2 + \frac{\mu^2}{\pi^2} \right). \tag{5.28}$$

($C_f = (N^2 - 1)/2N$ in SU(N) gauge theory, and $g^2 C_f \to e^2$ in QED.)

For frequencies $\omega < \hat{M}$, there are, in contrast to the gauge boson propagator, no solutions with imaginary wave-vectors that would correspond to screening. Instead, the additional collective ($-$) (occasionally dubbed "plasmino" [230]) branch exhibits propagating modes down to $\omega^{(-)}_{\mathrm{min.}} \approx 0.928\hat{M}$ (at $|\mathbf{k}|_{\mathrm{dip}} \approx 0.408\hat{M}$), with a curious dip in the dispersion curve reminiscent of that of rotons in liquid helium [231].

For momenta $|\mathbf{k}| \gg \hat{M}$, the normal ($+$) branch of the poles of the fermion propagator approaches asymptotically a mass hyperboloid with mass $M_\infty = \sqrt{2}\hat{M}$ and unit residue, whereas the ($-$) branch tends to the light-cone exponentially, with exponentially vanishing residue.

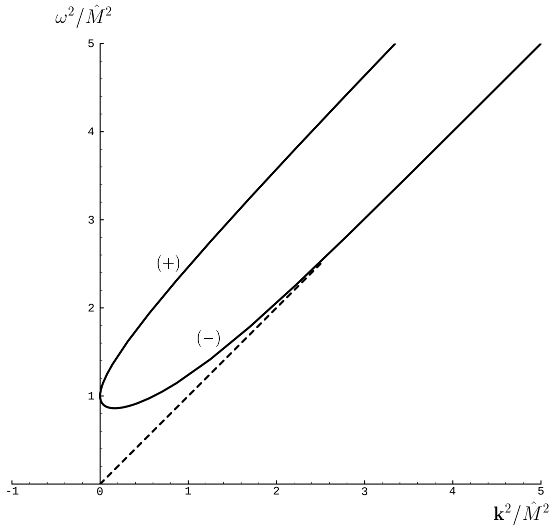


Figure 10. Location of poles in the hard-thermal-loop fermion propagator

For space-like momenta, $\omega^2 < \mathbf{k}^2$, there is again a large imaginary part $\propto \hat{M}^2$ corresponding to fermionic Landau damping, which now reflects the possibility of turning hard fermionic (bosonic) plasma constituents into hard bosonic (fermionic) ones in the presence of soft fermionic fields. Beyond lowest (HTL) order, this imaginary part cannot be expected to be gauge-fixing independent, however, not even in Abelian theories.

Another case where gauge dependences arise is when the bare fermion mass cannot be neglected compared to temperature. Then the HTL approximation is no longer adequate, and one has to include more than one-loop diagrams for consistency. At one-loop order, the modifications of the fermionic dispersion laws have been studied in references [232, 233], with the result that the additional $(-)$ branch disappears gradually when $m/T \gtrsim 1/3$. This phenomenon has been obtained both in Coulomb and Feynman gauge, with only weak gauge dependences in the real parts of the fermion self-energy, so it is not to be expected that higher-loop corrections would change this result qualitatively.

There have been also investigations of how the HTL dispersion laws of fermions as well as gauge bosons are affected by retaining non-leading powers of temperature [234, 235] and chemical potential [236, 237, 238, 239] which also emphasize the need for higher-loop contributions to obtain complete, gauge-independent results.

The very existence of the additional $(-)$ branch in the high-temperature limit has been confirmed in references [240, 241] under rather weak assumption without use of perturbation theory. Possible experimental signatures of the additional fermionic quasi-particles are the Van Hove singularities in dilepton production from a quark-gluon plasma [242, 243, 240] (though smeared out by damping effects to an unknown extent); they also play a role in the calculation of the electroweak baryon genesis of reference [244] but it has been shown [245, 246] that the fermion damping discussed in section 7.5 completely swamps their effects in the standard model (but may be still important in extensions thereof [247]).

5.3. Diquark condensates and colour superconductivity

At sufficiently low temperature and high quark chemical potential, QCD will in fact be in a colour superconducting phase which modifies the fermion propagator by the appearance of a diquark condensate. The reason is that even an arbitrarily weak attractive interaction at the Fermi surface leads to the appearance of Cooper pairs [248, 14], which form a Bose condensate and give rise to an energy gap leading to superfluidity, or, in the presence of a gauge symmetry which is spontaneously broken by the condensate, to superconductivity. Whereas in conventional superconductors, the only attractive interaction is from phonon-mediated interactions, in QCD already one-gluon exchange in the colour-antitriplet channel is attractive, so *colour*-superconductivity should be a much less fragile phenomenon with correspondingly large energy gaps. This was first studied by Barrois [249] in the late seventies and more extensively by Bailin and Love [250] and others [251, 252].

Over the last few years there has been a renewed flurry of activity in this field following the observation by various groups [253, 254] that the energy gap of colour superconductors as well as their critical temperature may be much larger than previously expected. Moreover, new symmetry breaking schemes were discovered. Whereas it was previously [250] thought that only the lightest quark flavours u and d are able to form Cooper pairs, it was argued in reference [253] and corroborated in reference [255] that a condensate which locks the breaking of colour and flavour symmetry ("colour-flavour locking" (CFL)) is energetically favoured, at least for high baryon chemical potential. In astrophysical applications, where in contrast to heavy-ion physics the strange-quark chemical potential is constrained by electric neutrality, CFL appears to be even more favoured than two-flavour superconductors (2SC) according to the estimates of [256, 257]. Alternatively, electric neutrality may be realized with the so-called gapless 2SC phase [258, 259].

Close to the phase transition to confined nuclear matter, there is also the possibility for rather involved symmetry breaking patterns. A particularly fascinating one is the so-called LOFF phase [260, 261, 262], which has a gap function that breaks translational invariance, leading to crystalline structures in a colour superconductor with possible relevance to neutron star physics [263].

Most of the original work has been based on a QCD inspired phenomenological Nambu-Jona-Lasinio (NJL) model [264]. Starting with the work of Son [265], the phenomenon of colour superconductivity has also been investigated by systematic perturbative techniques based on the fundamental Lagrangian of QCD. Son established, using renormalization group techniques, that in QCD the parametric form of the energy gap is not of the order of $b\mu\exp(-c/g^2)$ as with point-like interactions, but is modified by long-range colour magnetic interactions to the parametrically larger order of $b_1\mu g^{-5}\exp(-c_1/g)$, with $c_1 = 3\pi^2/\sqrt{2}$.

Using weak-coupling methods, b_1 has been calculated in [266, 267, 268, 269, 270, 271], and gauge parameter independence of b_1 has been verified in [272] in Coulomb-like gauges, though covariant gauges present a problem [273, 274, 275, 276].

The gauge independence identities of the previous section are not immediately applicable, but require a Nambu-Gor'kov ansatz [250, 271, 277] for the inverse propagator of the form

$$\mathcal{S}^{-1} = \begin{pmatrix} \not{q} + \mu\gamma_0 + \Sigma & \Phi^- \\ \Phi^+ & \not{q} - \mu\gamma_0 + \bar{\Sigma} \end{pmatrix}, \tag{5.29}$$

where $\Phi^\pm$ are the gap functions, related by $\Phi^-(q) = \gamma_0[\Phi^+(q)]^\dagger\gamma_0$, and $\bar{\Sigma}(q) =$

$C[\Sigma(-q)]^T C^{-1}$ with the charge conjugation matrix C. Flavour and fundamental colour indices are suppressed in (5.29). This inverse propagator is the momentum space version of the second derivative of the effective action,

$$\frac{\delta^2\Gamma}{\delta\bar{\Psi}(x)\delta\Psi(y)}\Big|_{\psi=\bar{\psi}=A_i^a=0,A_0^a=\tilde{A}_0^a}, \tag{5.30}$$

where $\Psi = (\psi, \psi_c)^T$, $\bar{\Psi} = (\bar{\psi}, \bar{\psi}_c)$, and $\tilde{A}_0^a$ is the expectation value of A_0^a, which is generally nonvanishing in the colour superconducting phase [278]. The doubling of fermionic fields in terms of Ψ and $\bar{\Psi}$ is just a notational convenience here; the effective action itself should be viewed as depending only on either $(\psi, \bar{\psi})$ or the set $\Psi = (\psi, \psi_c)^T$.

Assuming a spatially homogeneous quark condensate, one can then derive a gauge dependence identity for the momentum-space propagator of the form [278]

$$\delta\det(S^{-1}_{i\bar{j}}) + \delta\tilde{A}^{a0}\frac{\partial}{\partial\tilde{A}^{a0}}\det(S^{-1}_{i\bar{j}}) \equiv \delta_{\rm tot}\det(S^{-1}_{i\bar{j}}) = -\det(S^{-1}_{i\bar{j}})[\delta X^k_{,k} + \delta X^{\bar{k}}_{,\bar{k}}], \tag{5.31}$$

where the indices i and $\bar{i}$ comprise colour, flavour, Dirac and Nambu-Gor'kov indices. Just like in the case of ordinary spontaneous symmetry breaking, one has to consider a total variation [206, 207] of the determinant of the inverse quark propagator, with the first term corresponding to the explicit variation of the gauge fixing function, and the second term coming from the gauge dependence of $\tilde{A}_0^a$.

Since the determinant is equal to the product of the eigenvalues, equation (5.31) implies that the location of the singularities of the quark propagator is gauge independent, provided the singularities of $\delta X^k_{,k}$ do not coincide with those of the quark propagator. As above, one may argue that δX is 1PI up to a full ghost propagator, and up to gluon tadpole insertions, and the singularities of the ghost propagator are not correlated to the singularities of the quark propagator. Gauge independence of the zeros of the inverse fermion propagator then follows provided that also the 1PI parts of δX have no singularities coinciding with the singularities of the propagator.

At leading order, when the quark self energy can be neglected, this implies that the gap function is gauge independent on the quasiparticle mass shell (though at higher orders it becomes necessary to consider the complete dispersion relations of the quasiparticles).

6. Hard-thermal-loop effective action and resummation

As we have seen in several examples now, complete weak-coupling expansions in thermal field theory tend to require a reorganization of the standard loop expansion. Already the perturbative expansion of the thermal mass (3.8) of a scalar quasi-particle in Sect. 3.3 has shown that ordinary perturbation theory fails to determine higher-order corrections, but runs into infra-red problems. A resummation of the leading-order (HTL) mass is necessary (and sufficient in this case) to reorganize the perturbation series [64]. A thermal mass $\propto gT$ introduces an additional (soft for $g \ll 1$) mass scale, and whenever loop calculations receive important contributions from this scale, it is clearly mandatory to use propagators dressed by these masses.

In the general case, it is however equally important to include vertex corrections. In gauge theories, this is only natural as Ward identities tie up vertex functions with self-energies. But regardless of gauge symmetry considerations, if there are contributions to N-point one-loop vertex functions that are proportional to T^2 like the

HTL self-energies, they are as important as bare vertices when the external momentum scale is $\sim gT$. One then has

$$\Gamma^{\rm HTL}_{,N} \sim g^N T^2 k^{2-N} \sim g^{N-2} k^{4-N} \sim \frac{\partial^N \mathcal{L}_{\rm cl}}{\partial A^N}\Big|_{k\sim gT}, \tag{6.1}$$

for bosonic fields A.

A one-loop vertex function whose leading contribution (for soft external momenta) is proportional to a power of temperature greater than one is called HTL, as this is again dominated by a hard loop momentum $\sim T$. In fact, already in spinor QED there are infinitely many HTL vertex functions, namely those involving two external fermion lines and an arbitrary number of gauge bosons. In QCD, there are in addition HTL's with an arbitrary number of external gluons. These have been first identified in references [279, 213, 280] and used to set up a resummation programme for amplitudes involving soft external momenta [213, 281].

6.1. HTL effective action

In the case of scalar ϕ^4 theory, the only HTL is a mass term corresponding to a local HTL effective Lagrangian $\mathcal{L}^{\rm HTL}_{\rm scalar} = -\frac12 \hat m_{\rm th}^2 \phi^2$ with $\hat m^2_{\rm th}$ the leading-order term from (3.8). Remarkably, the infinitely many HTL diagrams of gauge theories have a comparatively simple and manifestly gauge-invariant integral representation [282, 283, 195, 284]

$$\begin{aligned}\mathcal{L}^{\rm HTL} &= \mathcal{L}^{\rm HTL}_f + \mathcal{L}^{\rm HTL}_g \\ &= \hat M^2 \int \frac{d\Omega_{\mathbf v}}{4\pi} \bar\psi \gamma^\mu \frac{v_\mu}{iv\cdot D(A)}\psi + \frac{\hat m_D^2}{2} {\rm tr} \int \frac{d\Omega_{\mathbf v}}{4\pi} F^{\mu\alpha} \frac{v_\alpha v^\beta}{(v\cdot D_{adj.}(A))^2} F_{\mu\beta}\end{aligned} \tag{6.2}$$

where $v = (1, \mathbf v)$ is a light-like 4-vector, i.e. with $\mathbf v^2 = 1$, and its spatial components are averaged over by $\int d\Omega_{\mathbf v} \cdots$. Here v is the remnant of the hard plasma constituents' momenta $p^\mu \sim T v^\mu$, namely their light-like 4-velocity, and the overall scale T has combined with the coupling constant to form the scale of the thermal masses, $\hat M, \hat m_D \sim gT$.

The covariant derivatives in the denominators of (6.2) are responsible for the fact that there are infinitely many HTL's. Because in QED one has $D_{adj.}(A) \to \partial$, the only HTL with exclusively photons as external lines is the photon self-energy polarization tensor; the other HTL diagrams of QED have two external fermion lines and an arbitrary number of photon insertions.

The gauge boson part of (6.2) has in fact been obtained originally in a form which is not obviously gauge invariant, namely

$$\mathcal{L}^{\rm HTL}_g = \hat m_D^2 \, {\rm tr} \left\{ A_0^2 + \int \frac{d\Omega_{\mathbf v}}{4\pi} \mathcal{W}(v\cdot A) \right\} \tag{6.3}$$

where the functional W is determined by being a gauge invariant completion of the Debye mass term [282] as

$$\mathcal{W}(v\cdot A) = \partial_0 (v\cdot A) F(\frac{1}{v\cdot\partial}[iv\cdot A, *]) \frac{1}{v\cdot \partial} v\cdot A, \quad F(z) = 2\sum_{n=0}^\infty \frac{z^n}{n+2}. \tag{6.4}$$

This has an interesting interpretation as eikonal of a Chern-Simons gauge theory [285, 286, 287], where however parity violations and a quantization of the coefficient of the action are absent because of the angular average.

Explicit representations of HTL vertices for gauge bosons are in fact most efficiently obtained from expanding (6.3) rather than the manifestly gauge invariant form (6.2). The gauge boson self energy (5.17) is then found to be given by

$$\hat{\Pi}_{\mu\nu}(k) = \hat{m}_D^2 \left[g_{\mu 0} g_{\nu 0} - k_0 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{v_\mu v_\nu}{v \cdot k} \right] \tag{6.5}$$

and an n-point vertex function by

$$\hat{\Gamma}^{a_1 \cdots a_n}_{\mu_1 \cdots \mu_n}(k_1, \cdots, k_n) = 2 g^{n-2} \hat{m}_D^2 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} v_{\mu_1} \cdots v_{\mu_n} \Big\{ \mathrm{tr}(T^{a_n}[T^{a_{n-1}}, [\ldots, T^{a_1}] \cdots]) \\ \times \frac{k_1^0}{v \cdot k_1} \frac{1}{v \cdot (k_1 + k_2)} \cdots \frac{1}{v \cdot (k_1 + \cdots + k_{n-2})} + \text{permutations}(1, \ldots, n-1) \Big\} \tag{6.6}$$

The effective Lagrangian (6.2) can be understood as an effective field theory in the sense of Wilson's renormalization group [288]. It arises from integrating out, in leading order, the effects of the hard momentum modes of the plasma constituents. Because, unlike the case of effective field theories in vacuum field theory, these particles are real rather than virtual, they give rise to non-localities corresponding to their free, light-like propagation. The HTL vertex functions encoded by (6.2) or (6.3) can thus be understood as forward scattering amplitudes for hard (collisionless) particles in soft external fields [195], thereby explaining their gauge independence which is less obvious on a purely diagrammatic level (there it can be understood through the absence of HTL ghost self-energy and vertex functions [192]).

The action provided by (6.2) or (6.3) is Hermitean only in a Euclidean form. After analytic continuation there are cuts corresponding to Landau damping, as mentioned above in connection with the two-point functions. In order to obtain the analytic continuation relevant for linear response theory, this is better performed on the effective equations of motions [289], which are also the primary objects in kinetic theory.

6.2. Kinetic theory approach

While a kinetic-theory derivation of the HTL propagators generalizing the Abelian case [214] to nonabelian Yang-Mills theory has been given already in references [290, 291, 292, 293], a systematic treatment that includes the HTL vertices has been developed only in references [294, 295, 284, 296, 297, 298] (see references [6, 299] for two recent comprehensive reviews). In a pure-glue nonabelian gauge theory, the effective equations of motion are given by

$$[D_\mu, F^{\mu\nu}]^a = j^{\nu a} = \hat{m}_D^2 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} v^\nu W^a(x, \mathbf{v}), \tag{6.7}$$

$$[v \cdot D, W(x, \mathbf{v})]^a = \mathbf{v} \cdot \mathbf{E}^a(x), \tag{6.8}$$

where $\mathbf{E}$ is the chromo-electric field strength, and $W^a(x, \mathbf{x})$ describes the fluctuations of the phase space density of hard gluons. At the expense of introducing $W^a(x, \mathbf{x})$ as a new soft degree of freedom, this formulation permits a completely local description of the physics of soft gauge boson modes [300, 301].

The HTL diagrams can be constructed from solving (6.7) in terms of $j^a_\mu[A]$ and formally expanding

$$j^a_\mu = -\hat{\Pi}^{ab}_{\mu\nu} A^\nu_b + \frac{1}{2} \hat{\Gamma}^{abc}_{\mu\nu\rho} A^\nu_b A^\rho_c + \ldots. \tag{6.9}$$

The kinetic-theory approach allows also to consider strong deviations from equilibrium. In references [302, 303, 304] the analogue of the HTL self-energies with anisotropic momentum distributions have been considered, which turn out to lead to space-like singularities in the gauge boson propagator. The latter are related to plasma instabilities which could play an important role in thermalization issues [305, 306].

6.3. HTL/HDL resummation

Since for soft momenta $\sim gT$ HTL self-energies and vertices are equally important as the tree-level self-energies and vertices, the former may not be treated perturbatively, but should rather be combined with the latter to form effective self-energies and vertices. This can be done formally by replacing

$$\mathcal{L}_{\rm cl} \to \mathcal{L}_{\rm cl} + \mathcal{L}^{\rm HTL} - \delta \times \mathcal{L}^{\rm HTL} \tag{6.10}$$

where δ is a parameter that is sent to 1 in the end, after the last term has been treated as a 'thermal counterterm' by assuming that δ counts as a one-loop quantity.

Because $\mathcal{L}^{\rm HTL}$ has been derived under the assumption of soft external momenta, this prescription is in fact only to be followed for soft propagators and vertices [213]. Propagators and vertices involving hard momenta (if present) do not require this resummation, and in fact for obtaining a systematic expansion in the coupling g, one has to expand out all HTL insertions on hard internal lines. In practice, a separation between hard and soft scales may be implemented by introducing an intermediate scale Λ with $gT \ll \Lambda \ll T$, assuming $g \ll 1$, for example $\Lambda \propto \sqrt{g}T$ (see e.g. [307, 308]).

The resulting systematic expansions in g typically involve single powers in g and logarithms of g, in contrast to conventional perturbation theory which would involve only g^2 as an expansion parameter. This is because increasing the loop order by one involves a factor g^2T which in an ultrarelativistic situation and soft external momenta $\lesssim gT$ is made dimensionless by a thermal mass $m \sim gT$, so that the effective expansion parameter becomes $g^2T/m \sim g$.

The same resummation scheme arises in the presence of a chemical potential. In the effective action, the chemical potential enters only in the mass parameters $\hat{M}^2$ and $\hat{m}_D^2$ according to (5.28) and (5.18). For $T \approx 0$ but large μ_f, the HTL have also been nicknamed "hard dense loops" (HDL) [309]. At $T \approx 0$, resummation of the HDL effective action is necessary for soft momenta $\lesssim g\mu$, but because of the absence of Bose enhancement this does not give rise to single powers of g in the perturbations series, but only to logarithms of g in addition to powers of g^2, as found long ago in the ring resummation scheme of Gell-Mann and Brueckner for an electron gas at high density [310]. HDL resummation for dynamic quantities has been considered first in [311] with applications to energy loss rates in astrophysical systems through axion-like particles, and for energy loss of heavy quarks at both large T and μ in [312].

HDL resummation is also at the basis of the results of reference [143] quoted in Sect. 4.4 on non-Fermi-liquid behaviour of the specific heat at low temperature and high chemical potential, equation (4.17). In this case the non-analytic terms in g^2 involve a logarithm of g as well as cubic roots of g^2, which are the result of the only weak dynamical screening of near-static magnetic modes whose screening lengths involve cubic roots of $m_D^2\omega$, see (5.19).

6.3.1. HTL-screened perturbation theory (HTLPT) In [106, 107, 108, 109] a modification of the above scheme has been suggested, where the mass parameters $\hat{m}_D$

and $\hat{M}$ within $\mathcal{L}^{\rm HTL}$ are considered as independent of the coupling g and used for a variational improvement of the perturbative series. This is a generalization of screened perturbation theory [100, 103]. It differs from standard HTL/HDL resummation also in that (6.10) is used for both hard and soft momenta, which results in additional UV divergences that need to be subtracted at any finite order of the expansion.

This method has been used to calculate the thermodynamic potential to two loop order, where it does improve the apparent convergence of the perturbative results, but the result deviates significantly from lattice results even at the highest temperatures that are available for the latter.

A problem of this approach, at least in quantities that are dominated by hard excitations like the pressure, seems to be that the HTL effective action is not a good approximation at hard momenta. This is also signalled by the fact that HTL propagators do not satisfy the relativistic KMS condition at large momenta [17]. While this is taken care of eventually by the counterterms in (6.10), at any finite order of the expansion there are uncancelled unphysical hard contributions. In [164] it has been shown that a (simpler) implementation of a variational perturbation theory in dimensional reduction which uses just the Debye mass term (which is the static limit of the HTL effective action) avoids this problem and indeed leads to results which are closer to the lattice result as well as the higher-order calculations in dimensional reduction when improved as discussed in section 4.2. It therefore appears that HTLPT needs to be amended such that a different treatment of hard and soft modes is secured.

6.3.2. HTL resummed thermodynamics through Φ-derivable approximations While HTLPT when applied to the thermodynamical potential does not work satisfactorily (though it may be of more use in quantities which depend more dominantly on soft rather than hard scales), it turns out that a generalization of the self-consistent expression for the entropy (3.18) allows for a resummation of HTL propagators without the problems of HTLPT and with remarkably good numerical results when compared to lattice data [96, 98, 99].¶

In gauge theories including fermions, the self-consistent two-loop expression for the entropy (3.18) reads

$$\begin{aligned}\mathcal{S} = & -\,\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}\frac{\partial n(\omega)}{\partial T}\left[\mathrm{Im}\,\log D^{-1}(\omega,k) - \,\mathrm{Im}\,\Pi(\omega,k)\,\mathrm{Re}\,D(\omega,k)\right] \\ & -2\,\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}\frac{\partial f(\omega)}{\partial T}\left[\mathrm{Im}\,\log S^{-1}(\omega,k) - \,\mathrm{Im}\,\Sigma(\omega,k)\,\mathrm{Re}\,S(\omega,k)\right], \qquad (6.11)\end{aligned}$$

and a similarly simple expression can be obtained for the quark number density

$$\mathcal{N} = -\,2\,\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}\frac{\partial f(\omega)}{\partial \mu}\left[\mathrm{Im}\,\log S^{-1}(\omega,k) - \,\mathrm{Im}\,\Sigma(\omega,k)\,\mathrm{Re}\,S(\omega,k)\right]. \qquad (6.12)$$

Here $n(\omega)=(e^{\beta\omega}-1)^{-1}$, $f(\omega)=(e^{\beta(\omega-\mu)}+1)^{-1}$, and "tr" refers to all discrete labels, including spin, colour and flavour when applicable.

In nonabelian gauge theories, the above expressions have to be augmented by Faddeev-Popov ghost contributions which enter like bosonic fields but with opposite over-all sign, unless a gauge is used where the ghosts do not propagate such as in axial gauges. But because Φ-derivable approximations do not generally respect

¶ A similar approach but formulated directly in terms of the (modified) thermodynamic potential Ω has been set up in [313].

gauge invariance,+ the self-consistent two-loop approximation will not be gauge-fixing independent. It is in fact not even clear that the corresponding gap equations (3.16) have solutions at all or that one can renormalize these (nonperturbative) equations, although nonperturbative renormalizability has been proven in the scalar case [92, 93, 94, 95]. Concerning gauge fixing dependences it is at least possible to show that at a stationary point these enter at twice the order of the truncation [316].

In [96, 98, 99] a manifestly gauge invariant approximation to full self-consistency has been proposed which maintains equivalence with conventional perturbation theory up to order g^3, which is the maximum (perturbative) accuracy of a two-loop Φ-derivable approximation. For these approximations it will be sufficient to keep only the two transverse structure functions of the gluon propagator and to neglect ghosts.

For soft momenta, the appropriate leading order propagator is the HTL one, and indeed there is no HTL ghost self-energy.

For hard momenta, one can identify the contributions to (6.11) below order g^4 as those linear in the self-energies,

$$\begin{aligned}\mathcal{S}^{\rm hard} &= \mathcal{S}_0 + 2N_g \int \frac{d^4k}{(2\pi)^4} \frac{\partial n}{\partial T} \,{\rm Re}\,\Pi_t \;{\rm Im}\, \frac{1}{\omega^2-k^2} \\ &\quad -4NN_f \int \frac{d^4k}{(2\pi)^4} \frac{\partial f}{\partial T} \left\{ {\rm Re}\,\Sigma_+ \,{\rm Im}\, \frac{-1}{\omega-k} \;-\; {\rm Re}\,\Sigma_- \,{\rm Im}\, \frac{-1}{\omega+k} \right\} \end{aligned} \tag{6.13}$$

considering now a gauge theory with N_g gluons and N_f fermion flavours. Because the imaginary parts of the free propagators restrict their contribution to the light-cone, only the light-cone projections of the self-energies enter. At order g^2 this is exactly given by the HTL results, without having to assume soft ω, k [317, 318]

$${\rm Re}\,\Pi_t^{(2)}(\omega^2=k^2) = \hat\Pi_t(\omega^2=k^2) = \tfrac12 \hat m_D^2 \equiv m_\infty^2, \tag{6.14}$$

$$2k\;{\rm Re}\,\Sigma_\pm^{(2)}(\omega=\pm k) = 2k\,\hat\Sigma_\pm(\omega=\pm k) = 2\hat M^2 \equiv M_\infty^2, \tag{6.15}$$

and without contributions from the other components of $\Pi_{\mu\nu}$ and the Faddeev-Popov self-energy.

There is no contribution $\propto g^2$ from soft momenta in (6.11) and (6.12) so that one is left with remarkably simple general formulae for the leading-order interaction contributions to the thermodynamic potentials expressed through the asymptotic thermal masses of the bosonic and fermionic quasiparticles:

$$\mathcal{S}^{(2)} = -T \left\{ \sum_B \frac{m_{\infty\,B}^2}{12} + \sum_F \frac{M_{\infty\,F}^2}{24} \right\}, \quad \mathcal{N}^{(2)} = -\frac{1}{8\pi^2} \sum_F \mu_F M_{\infty\,F}^2. \tag{6.16}$$

Here the sums run over all the bosonic (B) and fermionic (F) degrees of freedom (e.g. 4 for each Dirac fermion), which are allowed to have different asymptotic masses and, in the case of fermions, different chemical potentials.

The result (6.16) also makes it clear that relative-order-g corrections to m_∞^2, $M_\infty^2 \sim g^2T^2$ will contribute to the order-g^3 terms in $\mathcal{S}$ and $\mathcal{N}$.

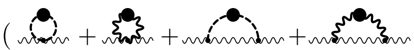
The HTL approximation to $\mathcal{S}$ and $\mathcal{N}$ thus includes correctly the leading-order interaction term $\propto g^2$ and only part of the order-g^3 terms. Using the peculiar sum rule (A.13) one can in fact show that in the case of pure-glue QCD the HTL entropy contains exactly 1/4 of the plasmon term $\sim g^3$. When treated strictly perturbatively, even 1/4 of the plasmon term spoils the apparent convergence.

+ For this, one would have to treat vertices on an equal footing with self-energies, which is in principle possible using the formalism developed in references [314, 139, 315].

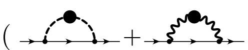
However, in the nonperturbative expression (6.11) the otherwise large g^3 correction is rendered harmless and leads to a small correction such that the rough agreement of the perturbative order-g^2 result with lattice results for $T \gtrsim 3T_c$ is retained and improved.

The plasmon term $\sim g^3$ becomes complete only upon inclusion of the next-to-leading correction to the asymptotic thermal masses m_∞ and M_∞. These are determined in standard HTL perturbation theory through

$$\delta m_\infty^2(k) = \mathrm{Re}\,\delta\Pi_T(\omega = k) \tag{6.17}$$

$= \mathrm{Re}\,($  $)|_{\omega=k}$

where thick dashed and wiggly lines with a blob represent HTL propagators for longitudinal and transverse polarizations, respectively. Similarly,

$\frac{1}{2k}\delta M_\infty^2(k) = \delta\Sigma_+(\omega = k) = \mathrm{Re}\,($  $)|_{\omega=k}\,.$ (6.18)

The explicit proof that these contributions indeed restore the correct plasmon term is given in reference [99].

These corrections to the asymptotic thermal masses are, in contrast to the latter, nontrivial functions of the momentum, which can be evaluated only numerically. However, as far as the generation of the plasmon term is concerned, these functions contribute in the averaged form

$$\bar{\delta} m_\infty^2 = \frac{\int dk\, k\, n'_{\mathrm{BE}}(k)\,\mathrm{Re}\,\delta\Pi_T(\omega = k)}{\int dk\, k\, n'_{\mathrm{BE}}(k)} \tag{6.19}$$

(cf. (6.13)) and similarly

$$\bar{\delta} M_\infty^2 = \frac{\int dk\, k\, n'_{\mathrm{FD}}(k)\,\mathrm{Re}\,2k\delta\Sigma_+(\omega = k)}{\int dk\, k\, n'_{\mathrm{FD}}(k)}\,. \tag{6.20}$$

These averaged asymptotic thermal masses turn out to be given by the remarkably simple expressions [99]

$$\bar{\delta} m_\infty^2 = -\frac{1}{2\pi} g^2 N T \hat{m}_D, \quad \bar{\delta} M_\infty^2 = -\frac{1}{2\pi} g^2 C_f T \hat{m}_D, \tag{6.21}$$

where $C_f = N_g/(2N)$. Since the integrals in (6.19) and (6.20) are dominated by hard momenta, these thermal mass corrections only pertain to hard excitations.

Pending a full evaluation of the NLO corrections to $\mathrm{Re}\,\delta\Pi$ and $\mathrm{Re}\,\delta\Sigma$, references [96, 98, 99] have proposed to define a next-to-leading approximation through (for gluons)

$$\mathcal{S}_{NLA} = \mathcal{S}_{HTL}\Big|_{\mathrm{soft}} + \mathcal{S}_{HTL, m_\infty^2 \to \bar{m}_\infty^2}\Big|_{\mathrm{hard}}, \tag{6.22}$$

where $\bar{m}_\infty^2$ includes (6.21) and a separation scale $\sqrt{c_\Lambda 2\pi T m_D}$ is introduced to make the distinction between hard and soft domains.

For numerical evaluations, a crucial issue here is the definition of the corrected asymptotic mass $\bar{m}_\infty$. For the range of coupling constants of interest ($g \gtrsim 1$), the correction $|\bar{\delta} m_\infty^2|$ is greater than the LO value m_∞^2, leading to tachyonic masses if included in a strictly perturbative manner.

However, this problem is not at all specific to QCD. In section 3.5 we have seen that the perturbative result (3.8) for the scalar screening mass to order g^3 also turns tachyonic for $g \gtrsim 1$. The self-consistent one-loop gap equation (3.6), on the other

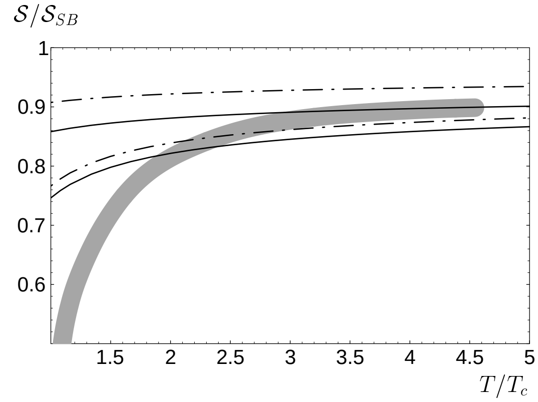


Figure 11. Comparison of the lattice data for the entropy of pure-glue SU(3) gauge theory of reference [154] (gray band) with the range of $\mathcal{S}_{HTL}$ (solid lines) and $\mathcal{S}_{NLA}$ (dash-dotted lines) for $\bar{\mu} = \pi T \ldots 4\pi T$ and $c_\Lambda = 1/2 \ldots 2$.

hand, is a monotonic function in g, and is well approximated by a quadratic equation (3.13) which keeps just to first two terms in in m/T expansion. As a model for the rather intractable (nonlocal) gap equations of QCD in a Φ-derivable approximation, reference [99] has therefore proposed to include the corrections (6.21) in analogy to (3.13) in order to enforce monotonicity of $\bar{m}_\infty$ as function of the coupling. (In order to be consistent with the large flavour number limit, this should in fact be done only for the bosonic part; for $N_f \neq 0$, the fermionic gap equation has to remain linear in $\bar{M}^2_\infty$, which however poses no problem at small N_f [319].)

In figure 11, the numerical results for the pure-glue HTL entropy and the NLA one (using a quadratic gap equation for $\bar{m}_\infty$) are given as a function of T/T_c with T_c chosen as $T_c = 1.14\Lambda_{\overline{\rm MS}}$. The full lines show the range of results for $\mathcal{S}_{HTL}$ when the renormalization scale $\bar{\mu}$ is varied from πT to $4\pi T$; the dash-dotted lines mark the corresponding results for $\mathcal{S}_{NLA}$ with the additional variation of c_Λ from 1/2 to 2. The dark-gray band are lattice data from reference [154]. Evidently, there is very good agreement for $T \gtrsim 2.5T_c$.

This approach can be generalized [98, 99, 171] also to nonzero chemical potentials μ_f, for which lattice results are available in the form of quark number susceptibilities [320, 321, 322, 323, 324] and are beginning to become available for finite small chemical potential [173, 174, 175, 176, 177, 178, 179, 180]. Simpler quasiparticle models [160, 161] have already been used to extrapolate lattice data to finite chemical potential [162] and seem to work well when compared with the recent lattice results for nonzero chemical potential [325]. The HTL approach offers a possible refinement, which has been worked out in [326, 327].

7. Next-to-leading order corrections to the quasi-particle spectrum

In the simple scalar model of Sect. 3.3 and in the more complicated example of asymptotic (averaged) gluon and quark masses, eqs. (6.19) and (6.20), we have seen that HTL/HDL perturbation theory leads to next-to-leading order corrections of quasi-particle dispersion laws which are typically suppressed by a single power of g rather than g^2.

In the following, we shall review what is at present known about next-to-leading order corrections to the HTL quasi-particles in gauge theories. It will turn out that some corrections, namely screening lengths for frequencies below the plasma frequency and damping rates for moving excitations are even more enhanced by infrared effects, to an extent that they cannot be determined beyond the leading logarithmic term without nonperturbative input.

7.1. Long-wavelength plasmon damping

Historically, the first full-fledged application of HTL resummation techniques was the calculation of the damping constant of gluonic plasmons in the long-wavelength limit. In fact, the development of the HTL resummation methods was stimulated by the failure of conventional thermal perturbation theory to determine this quantity.

In bare perturbation theory, the one-loop long-wavelength plasmon damping constant had been found to be gauge-parameter dependent and negative definite in covariant gauges [216]. Since even the gauge-independent frameworks of the Vilkovisky-DeWitt effective action [328, 329] and the gauge-independent pinch technique [330] led to negative one-loop damping constants, this was sometimes interpreted as a signal of an instability of the quark-gluon plasma [331, 332]. Other authors instead argued in favour of ghost-free "physical" gauges such as temporal and Coulomb gauge, where the result turned out to be positive and seemingly gauge-independent in this class of gauges [196, 197].

It was pointed out in particular by Pisarski [333] that all these results at bare one-loop order were incomplete, as also implied by the arguments for gauge independence of the singularities of the gluon propagator of reference [191]. The appropriate resummation scheme was finally developed by Braaten and Pisarski in 1990 [281, 213] who first obtained the complete leading term in the plasmon damping constant by evaluating the diagrams displayed in figure 12 [334] with the result

$$\begin{aligned}\gamma(\mathbf{k}=0) &= \frac{1}{2\hat{\omega}_{\rm pl}} \mathrm{Im}\,\delta\Pi_A(k_0=\hat{\omega}_{\rm pl},\mathbf{k}=0) \\ &= +6.635\ldots\frac{g^2NT}{24\pi} = 0.264\sqrt{N}g\,\hat{\omega}_{\rm pl}. \end{aligned} \tag{7.1}$$

For $g \ll 1$ this implies the existence of weakly damped gluonic plasmons. In QCD ($N=3$), where for all temperatures of interest $g \gtrsim 1$, the existence of plasmons as identifiable quasi-particles requires that g is significantly less than about 2.2, so that the situation is somewhat marginal.

The corresponding quantity for fermionic quasi-particles has been calculated in [335, 336] with a comparable result: weakly damped long-wavelength fermionic quasi-particles in 2- or 3-flavour QCD require that g is significantly less than about 2.7.

These results have been obtained in Coulomb gauge with formal verification of their gauge independence. Actual calculations, however, later revealed that in covariant gauges HTL-resummed perturbation theory still leads to explicit gauge dependent contributions to the damping of fermionic [210] as well as gluonic [211] quasi-particles. But, as was pointed out subsequently in [212], these apparent gauge dependences are avoided if the quasi-particle mass-shell is approached with a general infrared cut-off (such as finite volume) in place, and this cut-off lifted only in the end. This procedure defines gauge-independent dispersion laws; the gauge dependent parts are found to pertain to the residue, which at finite temperature happens to be

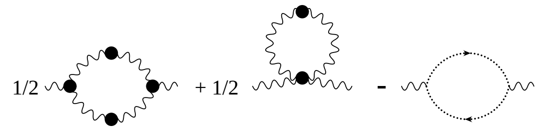


Figure 12. One-loop diagrams in HTL-resummed perturbation theory contributing to the gauge boson self-energy in pure-glue QCD. HTL-resummed quantities are marked with a blob.

linearly infrared singular in covariant gauges, rather than only logarithmically as at zero temperature, due to Bose enhancement.

For kinematical reasons, the result (7.1), which has been derived for branch A of the gluon propagator, should equally hold for branch B, since with $\mathbf{k} \to 0$ one can no longer distinguish between spatially transverse and longitudinal polarizations. However, as will be discussed further below, the limit $\mathbf{k} \to 0$ involves infrared problems, and there are even explicit calculations [337, 338] that claim to find obstructions to this equality, which are however contradicted by the recent work of [339].

In the simpler ϕ^4 theory the long-wavelength plasmon damping constant has been calculated in [64, 65] and it has been shown in [340, 341, 342, 343] that this quantity can also be extracted from classical field theory after perturbatively matching to the HTL mass.

7.2. NLO correction to the plasma frequency

In [344], Schulz has calculated also the real part of the NLO contribution to the gluon polarization tensor in the limit of $\mathbf{k} \to 0$ which determines the NLO correction to the gluonic plasma frequency.

The original power-counting arguments of [213] suggested that besides one-loop diagrams with HTL-resummed propagators and vertices given in figure 12, there could be also contributions from two-loop diagrams to relative order g. The explicit (and lengthy) calculation of [344] showed that those contribute only at order $g^2 \ln(1/g)$ rather than g, and the NLO plasma frequency in a pure-glue plasma was obtained as

$$\omega_{\rm pl.} = \hat\omega_{\rm pl.} \left[1 - 0.09\sqrt{N}g\right]. \tag{7.2}$$

In this particular result, HTL-resummed perturbation theory turns out to give a moderate correction to the leading-order HTL value even for $g \sim 1$.

7.3. NLO correction to the Debye mass

Poles of the gauge boson propagator at $\omega < \omega_{\rm pl.}$ and $\mathbf{k}^2 = -\kappa^2 < 0$ describe exponential screening of fields with frequencies below the plasma frequency and, for $\omega \to 0$, of static fields. The value of κ is the (frequency-dependent) screening mass.

In the static case, branch A of the propagator describes the screening of (chromo-) magnetostatic fields. While there is a finite screening length as long as $\omega > 0$, the A-branch of the HTL propagator becomes unscreened in the static limit. In QED, a "magnetic mass" is forbidden by gauge invariance [215, 222], but some sort of entirely non-perturbative magnetic mass is expected in non-Abelian gauge theories in view of severe infrared problems caused by the self-interactions of magnetostatic gluons [149, 150, 151].

Branch B, on the other hand, contains the information about screening of (chromo) electric fields as generated by static charges (Debye screening). The Debye mass given by the leading-order HTL propagator is $\hat{m}_D = \sqrt{3}\omega_{\rm pl.}$.

The evaluation of corrections in thermal perturbation theory again requires resummation. Even though the nonlocalities of the HTL effective action do not play a role in the static limit, historically these corrections have been determined for QCD, as well as for (ultra-relativistic) QED, only after the more complicated cases of gluonic plasmon damping and NLO plasma frequency discussed above had been mastered.

Originally, the Debye mass (squared) has been *defined* as the infrared limit $\Pi_{00}(\omega = 0, k \to 0)$, which indeed is correct at the HTL level, cf. (5.17).

In QED, this definition has the advantage of being directly related to the second derivative of the thermodynamic pressure with respect to the chemical potential μ (electric susceptibility), so that the higher-order terms known from the latter determine those of $\Pi_{00}^{\rm QED}(\omega = 0, k \to 0)$ through [215, 8]

$$\Pi_{00}(0, k \to 0)\Big|_{\mu=0} = e^2 \frac{\partial^2 P}{\partial \mu^2}\Big|_{\mu=0} = \frac{e^2 T^2}{3}\left(1 - \frac{3e^2}{8\pi^2} + \frac{\sqrt{3}e^3}{4\pi^3} + \dots\right) \tag{7.3}$$

This result is gauge independent because in QED all of $\Pi_{\mu\nu}$ is.

In the case of QCD, there is no such relation. Moreover, $\delta m_D^2/\hat{m}_D^2 \sim g$ rather than g^3 because of gluonic self-interactions and Bose enhancement. The calculation of this quantity should be much easier than the dynamic ones considered above, because in the static limit the HTL effective action collapses to just the local, bilinear HTL Debye mass term,

$$\mathcal{L}^{\rm HTL} \overset{\rm static}{\longrightarrow} -\frac{1}{2}\hat{m}_D^2 {\rm tr} A_0^2. \tag{7.4}$$

This is also gauge invariant, because A_0 behaves like an adjoint scalar under time-independent gauge transformations. Resummed perturbation theory for static quantities thus reduces to a resummation of the HTL Debye mass in the electrostatic propagator [310, 8, 114].

However, in QCD this simple ("ring") resummation leads to the gauge dependent result [345]

$$\Pi_{00}(0,0)/\hat{m}_D^2 = 1 + \alpha \frac{N}{4\pi}\sqrt{\frac{6}{2N + N_f}}\, g \tag{7.5}$$

where α is the gauge parameter of general covariant gauge (which coincides with general Coulomb gauge in the static limit).

This result was initially interpreted as meaning either that the non-Abelian Debye mass could not be obtained in resummed perturbation theory [346] or that one should use a physical gauge instead [196, 8]. In particular, temporal axial gauge was put forward, because in this gauge there is, like in QED, a linear relationship between electric field strength correlators and the gauge propagator. However, because static ring resummation clashes with temporal gauge, inconclusive and contradicting results were obtained by different authors [347, 348, 196]. A consistent calculation in fact requires vertex resummations [349, 350], but this does not resolve the gauge dependence issue because the nonabelian field strength correlator is gauge variant [351].

On the other hand, in view of the gauge dependence identities discussed in section 5.1.1, the gauge dependence of (7.5) is no longer surprising. Gauge independence can only be expected "on-shell", which here means $\omega = 0$ but $\mathbf{k}^2 \to -\hat{m}_D^2$.

Indeed, the exponential fall-off of the electrostatic propagator is determined by the position of the singularity of $\Delta_B(0,k)$, and not simply by its infrared limit. This implies in particular that one should use a different definition of the Debye mass already in QED, despite the gauge independence of (7.3), namely [352]

$$m_D^2 = \Pi_{00}(0,k)\Big|_{\mathbf{k}^2\to -m_D^2}. \tag{7.6}$$

For QED (with massless electrons), the Debye mass is thus not given by (7.3) but rather as [352]

$$\begin{aligned} m_D^2 &= \Pi_{00}(0,k\to 0) + \left[\Pi_{00}(0,k)\big|_{k^2=-m_D^2} - \Pi_{00}(0,k\to 0)\right] \\ &= \frac{e^2T^2}{3}\left(1 - \frac{3e^2}{8\pi^2} + \frac{\sqrt{3}e^3}{4\pi^3} + \ldots - \frac{e^2}{6\pi^2}[\ln\frac{\tilde\mu}{\pi T} + \gamma_E - \frac{4}{3}] + \ldots\right) \end{aligned} \tag{7.7}$$

where $\tilde\mu$ is the renormalization scale of the momentum subtraction scheme i.e. $\Pi_{\mu\nu}(k^2 = -\tilde\mu^2)|_{T=0} = 0$. In the also widely used modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme the last coefficient $-\frac{4}{3}$ in (7.7) has to be replaced by $-\frac{1}{2}$. (The slightly different numbers in the terms $\propto e^4T^2$ quoted in [222, 10] pertain to the minimal subtraction (MS) scheme*.)

Since $de/d\ln\tilde\mu = e^3/(12\pi^2) + O(e^5)$, (7.7) is a renormalization-group invariant result for the Debye mass in hot QED, which (7.3) obviously is not. Only the susceptibility $\chi = \partial^2P/\partial\mu^2$ is renormalization-group invariant, but not $e^2\chi$. Similarly, one should distinguish between electric susceptibility and electric Debye mass also in the context of QCD.

In QCD, where gauge independence is not automatic, the dependence on the gauge fixing parameter α is another indication that (7.5) is the wrong definition. For (7.6) the full momentum dependence of the correction $\delta\Pi_{00}(k_0 = 0,\mathbf{k})$ to $\hat\Pi_{00}$ is needed. This is given by [352]

$$\begin{aligned} \delta\Pi_{00}(k_0 = 0,\mathbf{k}) &= g\hat m_D N\sqrt{\frac{6}{2N+N_f}}\int\frac{d^{3-2\varepsilon}p}{(2\pi)^{3-2\varepsilon}}\bigg\{\frac{1}{\mathbf{p}^2+\hat m_D^2} + \frac{1}{\mathbf{p}^2} \\ &+ \frac{4\hat m_D^2 - (\mathbf{k}^2+\hat m_D^2)[3+2\mathbf{pk}/\mathbf{p}^2]}{\mathbf{p}^2[(\mathbf{p}+\mathbf{k})^2+\hat m_D^2]} + \alpha(\mathbf{k}^2+\hat m_D^2)\frac{\mathbf{p}^2+2\mathbf{pk}}{\mathbf{p}^4[(\mathbf{p}+\mathbf{k})^2+\hat m_D^2]}\bigg\}. \end{aligned} \tag{7.8}$$

In accordance with the gauge dependence identities, the last term shows that gauge independence holds algebraically for $\mathbf{k}^2 = -\hat m_D^2$. On the other hand, on this "screening mass shell", where the denominator term $[(\mathbf{p}+\mathbf{k})^2+\hat m_D^2] \to [\mathbf{p}^2+2\mathbf{pk}]$, one encounters IR-singularities. In the α-dependent term, they are such that they produce a factor $1/[\mathbf{k}^2+m_D^2]$ so that the gauge dependences no longer disappear even on-shell. This is, however, the very same problem that had to be solved in the above case of the plasmon damping in covariant gauges. Introducing a temporary infrared cut-off (e.g., finite volume), does not modify the factor $[\mathbf{k}^2+m_D^2]$ in the numerator but removes the infrared divergences that would otherwise cancel it. Gauge independence thus holds for all values of this cut-off, which can be sent to zero in the end. The gauge dependences are thereby identified as belonging to the (infrared divergent) residue.

The third term in the curly brackets, however, remains logarithmically singular on-shell when the infrared cut-off is removed. In contrast to the α-dependent term, closer inspection reveals that these singularities are coming from the massless magnetostatic modes and not from unphysical massless gauge modes.

* Reference [10] erroneously refers to the $\overline{\mathrm{MS}}$ scheme when quoting the MS result of [222].

At HTL level, there is no (chromo) magnetostatic screening, but, as we have mentioned, one expects some sort of such screening to be generated non-perturbatively in the static sector of hot QCD at the scale $g^2T \sim gm_D$ [149, 150, 151].

While this singularity prevents evaluating $\delta\Pi_{00}$ in full, the fact that this singularity is only logarithmic allows one to extract the leading term of (7.8) under the assumption of an effective cut-off at $p \sim g^2T$ as [352]

$$\frac{\delta m_D^2}{\hat{m}_D^2} = \frac{N}{2\pi}\sqrt{\frac{6}{2N+N_f}}\, g\, \ln\frac{1}{g} + O(g). \tag{7.9}$$

The $O(g)$-contribution, however, is sensitive to the physics of the magnetostatic sector at scale g^2T, and is completely non-perturbative in that all loop orders ≥ 2 are expected to contribute with equal importance as we shall discuss in section 7.4.

Because of the undetermined $O(g)$-term in (7.9), one-loop resummed perturbation theory only says that for sufficiently small g, where $O(g\ln(1/g)) \gg O(g)$, there is a *positive* correction to the Debye mass of lowest-order perturbation theory following from the pole definition (7.6), and that it is gauge independent.

On the lattice, the static gluon propagator of pure SU(2) gauge theory at high temperature has been studied in various gauges [353, 223] with the result that the electrostatic propagator is exponentially screened with a screening mass that indeed appears to be gauge independent and which is about 60% larger than the leading-order Debye mass for temperatures T/T_c up to about 10^4. Similar results have been obtained recently also for the case of SU(3) [354, 355].

In [351], an estimate of the $O(g)$ contribution to (7.9) has been made using the crude approximation of a simple massive propagator for the magnetostatic one, which leads to

$$\frac{\delta m_D^2}{\hat{m}_D^2} = \frac{N}{2\pi}\sqrt{\frac{6}{2N+N_f}}\, g\, \left[\ln\frac{2m_D}{m_m} - \frac{1}{2}\right]. \tag{7.10}$$

On the lattice one finds strong gauge dependences of the magnetostatic screening function, but the data are consistent with an over-all exponential behaviour corresponding to $m_m \approx 0.5g^2T$ in all gauges [353, 356]. Using this number in a self-consistent evaluation of (7.10) gives an estimate for m_D which is about 20% larger than the leading-order value for $T/T_c = 10\ldots10^4$.

This shows that there are strong non-perturbative contributions to the Debye screening mass m_D even at very high temperatures. Assuming that these are predominantly of order g^2T, one-loop resummed perturbation theory (which is as far as one can get) is able to account for about 1/3 of this inherently non-perturbative physics already, if one introduces a simple, purely phenomenological magnetic screening mass.

7.3.1. Non-perturbative definitions of the Debye mass A different approach to studying Debye screening non-perturbatively without the complication of gauge fixing is to consider spatial correlation functions of appropriate gauge-invariant operators such as those of the Polyakov loop

$$L(\mathbf{x}) = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\left\{-ig\int_0^\beta d\tau\, A_0(\tau,\mathbf{x})\right\}. \tag{7.11}$$

The correlation of two such operators is related to the free energy of a quark-antiquark pair [357]. In lowest order perturbation theory this is given by the square of a Yukawa

potential with screening mass $\hat{m}_D$ [346]; at one-loop order one can in fact identify contributions of the form (7.10) if one assumes magnetic screening [358, 351], but there is the problem that through higher loop orders the large-distance behaviour becomes dominated by the magnetostatic modes and their lightest bound states [359].

In [360], Arnold and Yaffe have proposed to use Euclidean time reflection symmetry to distinguish electric and magnetic contributions to screening, and have given a prescription to compute the sublogarithmic contribution of order g^2T to m_D nonperturbatively. This has been carried out in 3-d lattice simulations for SU(2) [361, 362] as well as for SU(3) [363]. The Debye mass thus defined shows even larger deviations from the lowest-order perturbative results than that from gauge-fixed lattice propagators. E.g., in SU(2) at $T = 10^4 T_c$ this deviation turns out to be over 100%, while in SU(3) the dominance of g^2T contributions is even more pronounced.

Clearly, (resummed) perturbation theory is of no use here for any temperature of practical interest. However, it should be noted that the magnitude of the contributions from the completely nonperturbative magnetostatic sector depends strongly on the quantity considered. It is significantly smaller in the definition of the Debye mass through the exponential decay of gauge-fixed gluon propagators, which leads to smaller screening masses on the lattice [353, 223, 354, 355] and which also seem to provide a useful nonperturbative description of Debye screening as they turn out to be gauge-independent in accordance with the arguments of section 5.1.1.

7.4. Magnetostatic screening

The next-to-leading order correction to the other structure function, $\Pi_{ii}(0,k)$, can be derived in a calculation analogous to the one leading to (7.8), and is found to be [352]

$$\delta\Pi_{ii}(0,k) = g^2 N \hat{m}_D T \left\{ \frac{(\alpha+1)^2+10}{16} \frac{k}{\hat{m}_D} + \frac{1}{4\pi}\left[2 - \frac{k^2+4\hat{m}_D^2}{\hat{m}_D k} \arctan\frac{k}{2\hat{m}_D} \right] \right\}. \tag{7.12}$$

Apparently, the magnetic permeability defined by $1/\mu = 1 - \frac{1}{2}\Pi_{ii}/k^2$ is a gauge-dependent quantity beyond leading order. The gauge-dependent terms vanish only at the location of the pole of the transverse gluon propagator, which is at $\mathbf{k} = 0$. There the correction term vanishes completely, which means that there is no magnetic mass squared of the order g^3T^2. The magnetic mass must therefore be $\ll g^{3/2}T$ at weak coupling.

For small $k \ll \hat{m}_D$, (7.12) has a linear behaviour with gauge dependent, but positive definite coefficient. The transverse propagator therefore has the form $1/(k^2 - ck)$ with $c \sim g^2T$. This corresponds to a pole at space-like momentum with $k \sim g^2T$ [364]. However, in this regime the inherently nonperturbative contributions $\sim g^4T^2$ to $\Pi_{\mu\nu}$ become relevant and are expected to remove this pathology by the generation of a magnetic mass $m_m \sim g^2T$. This is completely analogous to the infrared problem of the perturbative expansion of the pressure identified in [150, 151], but while the latter sets in at 4-loop order, in the transverse gauge-boson self-energy with external momenta $\lesssim g^2T$ this problem starts already at 2-loop order.

In a HTL-resummed 2-loop calculation in general covariant gauge, reference [365] has verified that such a magnetic mass receives contributions exclusively from the ultrasoft momentum regime $k \sim g^2T$ where the relevant effective theory is three-dimensional Yang-Mills theory, which is confining and inherently nonperturbative.

There exists in fact an elegant attempt towards an analytic nonperturbative study of this theory through its Schrödinger equation using results from two-dimensional conformal field theory [366, 367, 368]. This leads to the estimate $m_m = g^2 NT/(2\pi)$, which is reasonably close to the results for magnetostatic propagators obtained in the lattice calculations of references [353, 223] quoted above.

The other approaches that have been tried to obtain an analytic estimate for the magnetic mass mostly involve a reorganization of perturbation theory by assuming a more or less complicated mass term for the magnetostatic gluons, setting up a self-consistent gap equation and solving it. However, the various possibilities lead to rather contradictory results [369, 370, 371, 372, 373, 374]. In fact, lattice calculations indicate that the phenomenon of magnetic screening is not well described by a simple pole mass [223].

In the Abelian case, one may expect that magnetostatic fields are completely unscreened and it can indeed be proved rigorously to all orders of perturbation theory [215, 222] that $\Pi_{ii}(0, k \to 0) = O(k^2)$. In massless scalar electrodynamics, an unresummed one-loop calculation actually gives $\Pi_{ii}(0, k \to 0) = \frac{1}{8}e^2 kT$ suggesting that the magnetostatic propagator has the form $1/[k^2 + e^2 kT/8]$, which would imply power-law screening. However, a resummation of the thermal mass of the scalar particles removes this unphysical behaviour [375, 222].

7.5. Dynamical screening and damping at high temperature

A logarithmic sensitivity to the nonperturbative physics of the (chromo-) magnetostatic sector has in fact been encountered early on also in the calculation of the damping rate for a heavy fermion [333], and more generally of hard particles [376, 377, 378, 379]. It also turns out to occur for soft quasi-particles as soon as they have nonvanishing (group) velocity [380, 381].

Because this logarithmic sensitivity arises only if one internal line of (resummed) one-loop diagrams is static, the coefficient of the resulting $g \ln(1/g)$-term is almost as easy to obtain as in the case of the Debye mass, even though the external line is non-static and soft, therefore requiring HTL-resummed vertices (see figure 12).

The infrared singularity arises (again) from the dressed one-loop diagram with two propagators, one of which is magnetostatic and thus massless in the HTL approximation, and the other of the same type as the external one, so only the first diagram in figure 12 is relevant. The dressed 3-vertices in it are needed only in the limit of one leg being magnetostatic and having zero momentum. Because of the gauge invariance of HTL's, these are determined by the HTL self-energies through a differential Ward identity, e.g.

$$\hat\Gamma_{\mu\nu\varrho}(k; -k; 0) = -\frac{\partial}{\partial k^\varrho}\hat\Pi_{\mu\nu}(k) \tag{7.13}$$

for the 3-gluon vertex (colour indices omitted).

Comparatively simple algebra gives [381]

$$\delta\Pi_I(k) \simeq -g^2 N 4\mathbf{k}^2[1 + \partial_{\mathbf{k}^2}\Pi_I(k)]^2 \mathcal{S}_I(k), \qquad I = A, B \tag{7.14}$$

where

$$\mathcal{S}_I(k) := T\int \frac{d^3p}{(2\pi)^3}\frac{1}{\mathbf{p}^2}\frac{-1}{(k-p)^2 - \Pi_I(k-p)}\Big|_{k^2=\Pi_I(k), p^0=0} \tag{7.15}$$

and the logarithmic (mass-shell) singularity arises because $(k-p)^2 - \Pi_I(k-p) \to -\mathbf{p}^2 + 2\mathbf{pk} - \Pi_I(k-p) + \Pi_I(k) \sim |\mathbf{p}|$ as $k^2 \to \Pi_I(k)$.

The IR-singular part of $\mathcal{S}_I(k)$ is given by

$$\begin{aligned}\mathcal{S}_I(k) &= T\int \frac{d^3p}{(2\pi)^3}\frac{1}{\mathbf{p}^2}\frac{1}{\mathbf{p}^2-2\mathbf{pk}+\Pi_I(k-p)-\Pi_I(k)-\mathrm{i}\varepsilon}\\ &\simeq T[1+\partial_{\mathbf{k}^2}\Pi_I(k)]^{-1}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\mathbf{p}^2}\frac{1}{\mathbf{p}^2-2\mathbf{pk}-\mathrm{i}\varepsilon}\\ &= T[1+\partial_{\mathbf{k}^2}\Pi_I(k)]^{-1}\int_\lambda^\infty\frac{dp}{p}\frac{1}{2|\mathbf{k}|}\ln\frac{p+2|\mathbf{k}|-\mathrm{i}\varepsilon}{p-2|\mathbf{k}|-\mathrm{i}\varepsilon}\end{aligned}\tag{7.16}$$

where in the last line we have inserted an IR cutoff $\lambda \ll gT$ for the p-integral in order to isolate the singular behaviour.

One finds that (7.16) has a singular imaginary part for propagating modes, and a singular real part in screening situations

$$\mathcal{S}_I(k)\simeq\frac{T}{8\pi}[1+\partial_{\mathbf{k}^2}\Pi_I(k)]^{-1}\times\begin{cases}\mathrm{i}|\mathbf{k}|^{-1}\ln(|\mathbf{k}|/\lambda)+O(\lambda^0) & \text{for } \mathbf{k}^2>0\\ \kappa^{-1}\ln(\kappa/\lambda)+O(\lambda^0) & \text{for } \mathbf{k}^2=-\kappa^2<0\end{cases}\tag{7.17}$$

The case $\mathbf{k}=0$, on the other hand, is IR-safe, because (7.14) is proportional to $\mathbf{k}^2$, while

$$\mathcal{S}_I(k)\longrightarrow\frac{T}{4\pi^2\lambda}[1+\partial_{\mathbf{k}^2}\Pi_I(k)]^{-1}\Big|_{\mathbf{k}=0}+O\left(\frac{T|\mathbf{k}|}{\lambda^2}\right)\quad\text{for } \mathbf{k}\to 0.\tag{7.18}$$

This shows that there is a common origin for the infrared sensitivity of screening and damping of HTL quasi-particles. Provided that the scale where the logarithmic divergences are cut off is the magnetic scale g^2T, the coefficients of the leading $g\ln(1/g)$-terms are determined and in fact lead to beautifully simple results: For the damping of moving quasi-particles one obtains [380, 381]

$$\gamma_I(|\mathbf{k}|)\simeq\frac{g^2NT}{4\pi}\frac{|\mathbf{k}|[1+\partial_{\mathbf{k}^2}\Pi_I(k)]}{\omega(|\mathbf{k}|)[1-\partial_{\omega^2}\Pi_I(k)]}\ln\frac{1}{g}\equiv\frac{g^2NT}{4\pi}v_I(|\mathbf{k}|)\ln\frac{1}{g}\tag{7.19}$$

where $v_I(|\mathbf{k}|)$ is the group velocity of mode I (which vanishes at $\mathbf{k}=0$). The IR-sensitive NLO correction to screening takes its simplest form when formulated as [381]

$$\delta\kappa_I^2(\omega)=\frac{g^2NT}{2\pi}\kappa_I(\omega)\left(\ln\frac{1}{g}+O(1)\right)\tag{7.20}$$

where $\kappa_I(\omega)$ is the inverse screening length of mode I at frequency $\omega<\omega_{\mathrm{pl.}}$ (which in the static limit approaches the Debye mass and perturbatively vanishing magnetic mass, resp., while approaching zero for both modes as $\omega\to\omega_{\mathrm{pl.}}$).

A completely analogous calculation for the fermionic modes (for which there are no screening masses) gives

$$\gamma_\pm(|\mathbf{k}|)=\frac{g^2C_FT}{4\pi}\big|v_\pm(|\mathbf{k}|)\big|\left(\ln\frac{1}{g}+O(1)\right)\tag{7.21}$$

for $|\mathbf{k}|>0$. The group velocity $v_\pm$ equals $\pm\frac{1}{3}$ in the limit $\mathbf{k}\to 0$, and increases monotonically towards $+1$ for larger momenta (with a zero for the $(-)$-branch at $|\mathbf{k}_{\mathrm{dip}}|/\hat{M}\approx 0.41$). For strictly $|\mathbf{k}|=0$, the IR sensitivity in fact disappears because (7.21) is no longer valid for $|\mathbf{k}|\ll\lambda$, but one has $\gamma_\pm(|\mathbf{k}|)|_{\mathrm{sing.}}\propto g^2T|\mathbf{k}|/\lambda$ instead. Thus $\gamma_\pm(0)$ is calculable at order g^2T in HTL-resummed perturbation theory, and has been calculated in [335, 336]. For nonvanishing $|\mathbf{k}|\sim\lambda\sim g^2T$ HTL-resummed perturbation theory breaks down (the IR sensitivity of $\gamma_\pm$ for small nonvanishing $\mathbf{k}$ has recently been displayed also in [382]). For $|\mathbf{k}|\gtrsim gT$, the damping is calculable in

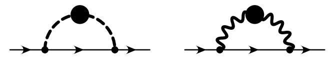


Figure 13. Next-to-leading order contributions to the self-energy of a hard fermion. The dashed and wiggly lines with a blob refer to the HTL/HDL resummed longitudinal and transverse gauge boson propagators, respectively.

the leading-log approximation, and even completely at $|\mathbf{k}| = |\mathbf{k}_{\rm dip}|$ in the case of γ_-, though this has not been done yet.

The fermionic result (7.21) applies in fact equally to QED, for which one just needs to replace $g^2 C_F \to e^2$. This is somewhat disturbing as QED does not allow a non-zero magnetic mass as IR cutoff, and it has been conjectured that the damping $\gamma \sim g^2 T$ or $e^2 T$ itself might act as an effective IR cutoff [376, 377, 380, 383], which however led to further difficulties [384, 385]. The solution for QED was finally found in references [386, 387, 388, 389] where it was shown that there the fermionic modes undergo over-exponential damping in the form $\mathrm{e}^{-\gamma t} \to \exp(-\frac{e^2}{4\pi} T t \ln(\omega_{\rm pl.} t))$ (for $v \to 1$), so finite time is the actual IR cut-off (see also [390, 391, 392]). The fermion propagator has in fact no simple quasi-particle pole in momentum space or any other singularity near the light-cone [190], but nevertheless a sharply peaked spectral density.

In non-Abelian theories, on the other hand, one does expect static (chromo-)-magnetic fields to have finite range, and lattice results do confirm this, so the above leading-log results for an (exponential) damping constant are expected to be applicable after all, at least for sufficiently weak coupling.

As will be discussed further in section 8, the damping rate of hard gluons $\gamma = \gamma_A(|\mathbf{k}| \sim T) \simeq \frac{1}{4\pi} g^2 N T \ln(1/g)$ defines an important scale in the dynamics of nonabelian fields that is parametrically larger than the magnetic mass scale $g^2 T$. It sets the time scale for colour relaxation and determines, to leading order, the colour conductivity through [393, 394, 395]

$$\sigma_c = \hat{\omega}_{\rm pl}^2 / \gamma. \qquad (7.22)$$

7.6. Damping of high-momentum fermions in a degenerate plasma

At zero temperature and nonvanishing chemical potential, the damping rate of fermionic excitations is calculable in perturbation theory, for in the absence of Bose enhancement the dynamical screening of the quasi-static transverse modes in the HDL gauge boson propagator is sufficient to remove all infrared divergences [396, 397, 398]. The leading-order damping rate of a high energy/momentum mode with $E, p \gg g\mu$ is given by the diagrams in figure 13, where only the gauge boson propagator needs to be resummed. Explicitly, one has

$$\begin{aligned} \gamma_+(E) = \frac{g^2 C_F}{2v} \int \frac{d^3 q}{(2\pi)^3} \left[\theta(q_0) - \theta(\mu - E + q_0)\right] \\ \times \left[\hat{\rho}_\ell(q_0, q) + v^2 (1 - \cos^2\xi) \hat{\rho}_t(q_0, q)\right]_{q_0 = qv\cos\xi} \end{aligned} \qquad (7.23)$$

where $v = p/E$ is the velocity of the fermion and $\hat{\rho}_{t,\ell}$ are defined in (A.1) and (A.2). (In the case of QED $g^2 C_F$ reduces simply to e^2.) For massless fermions $v = 1$ and $\hat{\rho}_{t,\ell}$ are the HDL expressions given in appendix A.1, but (7.23) remains applicable to massive fermions and correspondingly modified spectral functions for the gauge bosons [309, 398].

Since dynamical screening vanishes in the zero-frequency limit according to (5.19), there is a qualitative deviation from the usual behaviour of the damping of fermionic excitations in the vicinity of the Fermi surface, which for short-range interactions vanishing like $(E-\mu)^2$ as $E \to \mu$ due to phase-space restrictions in the fermion-fermion scattering [399]. The magnetic interactions instead give rise to a $|E-\mu|$ behaviour characteristic of non-Fermi liquids [183, 396, 397, 398] which reads

$$\gamma_+(E) = \frac{g^2 C_F}{24\pi}|E-\mu| + \frac{g^2 C_F}{64\, v_F^2 m_D}(E-\mu)^2 + \mathrm{O}(|E-\mu|^3) \tag{7.24}$$

with $m_D^2 = v_F \hat{m}_D^2$ and $\hat{m}_D^2$ given by the $T=0$ limit of (5.18). Here the first term is from the dynamically screened quasi-static magnetic interactions and the second term from Debye screened electrostatic ones. In a nonrelativistic situation ($v \ll 1$) the latter is the dominant one (except very close to the Fermi surface).

Far away from the Fermi surface, the damping approaches [397]

$$\gamma_+(E) \sim 0.019\, g^2 C_F m_D, \quad E-\mu \gg \mu \tag{7.25}$$

which is also the damping relevant for hard anti-fermions [398] (which are automatically far from their Fermi surface).

7.7. Non-Fermi-liquid contributions to the real part of the fermion self-energy

The real part of the fermion self-energy at zero temperature can equally be calculated in HDL perturbation theory, but the most interesting aspect of it, a nonanalytic behaviour in the vicinity of the Fermi surface, can be inferred from the behaviour of its imaginary part through a Kramers-Kronig dispersion relation, which implies [183, 270, 398, 400]

$$\mathrm{Re}\,\Sigma_+(E,p) \simeq \mathrm{Re}\,\Sigma_+(\mu,p) - \frac{g^2 C_F}{12\pi^2}(E-\mu)\ln\frac{m_D}{|E-\mu|} + \mathrm{O}(|E-\mu|). \tag{7.26}$$

This quantity is gauge independent on the mass shell of the hard particle, to which (7.26) presents the leading correction. The energy-independent part is governed by the asymptotic fermionic mass,

$$\mathrm{Re}\,\Sigma_+(\mu,p) = M_\infty^2/(2p), \quad M_\infty^2 = 2\hat{M}^2 = \frac{1}{4\pi^2} g^2 C_f \mu^2. \tag{7.27}$$

This corresponds to a correction to the Fermi momentum defined by $\omega_+(p_F) = \mu$, which for effectively massless fermions reads

$$p_F/\mu = 1 - \frac{g^2 C_f}{8\pi^2} + \ldots . \tag{7.28}$$

The logarithmic term in (7.26) leads to a correction to the group velocity of the form

$$v_g = \frac{\partial E}{\partial p} = 1 - \frac{g^2 C_f}{12\pi^2}\ln\frac{m_D}{|E-\mu|} + \ldots , \tag{7.29}$$

which dominates over the contribution from the asymptotic fermion mass for $|E-\mu| \ll m_D$, and eventually spoils (HDL) perturbation theory when $|E-\mu|/m_D \lesssim \exp(-C/g^2)$. Evidently, the quasistatic magnetic interactions lead to significant changes in the vicinity of the Fermi surface. Such non-Fermi-liquid corrections can even be the dominant effects in certain quantities, such as the entropy or specific heat at low temperature $T \ll m_D \sim g\mu$, as discussed in section 4.4. In a colour superconductor, where the quasiparticle and quasiholes at the Fermi surface develop a

gap of order $b_1 \mu g^{-5} \exp(-c_1/g)$ as mentioned in section 5.3, the non-Fermi-liquid corrections to the real part of the fermion self-energy remain perturbative and contribute at the level of the constant b_1, resulting in a significant reduction of the magnitude of the gap [269, 271] compared to previous results which did not include non-Fermi-liquid contributions to the quark self-energy [266, 267, 268].

One should note that the nonanalytic behaviour of (7.26) is indeed in the energy variable rather than the momentum, despite the fact that the derivation has been on-shell where the two are related, as can be verified by explicit calculation of $\mathrm{Re}\,\partial\Sigma_+/\partial p_i$, which is indeed analytic on the Fermi surface [270]. One consequence of this is that similar logarithmic terms appear in the quark-gluon vertex only in a special kinematic regime, namely in

$$\lim_{E' \to E \approx \mu} \lim_{p' \to p \approx \mu} \Lambda_\mu(E', \mathbf{p}'; E, \mathbf{p}) = \frac{g^3 C_f}{12\pi^2} \delta^0_\mu \ln \frac{m_D}{|E-\mu|} + \mathrm{O}(|E-\mu|), \tag{7.30}$$

where $(E', \mathbf{p}')$ and $(E, \mathbf{p})$ are the four-momenta of the quarks, but not when the order of the limits is interchanged. Since magnetic interactions become (almost) unscreened only in the regime where $\omega \ll k$, it can been argued that non-Fermi-liquid corrections manifest themselves only through corrections on the quark self-energy rather than the vertices [270].

7.8. NLO corrections to real parts of dispersion laws at high temperature

At nonvanishing temperature, the real parts of the dispersion laws of fermionic and gluonic quasi-particles remain IR-safe in NLO HTL-resummed perturbation theory (in contrast to the imaginary parts which are perturbatively accessible only at $T = 0$ or for $\mathbf{k} = 0$ and exceptional momenta where the group velocity vanishes). However, such calculations are exceedingly involved, and only some partial results exist so far in QCD [401, 402].

In the following, we shall restrict our attention to the case $\mathbf{k}^2/\omega^2_{\mathrm{pl.}} \gg 1$ and consider the different branches of the dispersion laws in turn.

7.8.1. Energetic quarks and transverse gluons At large momenta only the normal branch of quark excitations and the transverse gluons have nonnegligible spectral weight. Their leading-order thermal masses become momentum independent in this limit and are given in one-loop order by their HTL values, $m^2_\infty = \hat{m}^2_D/2$ and $M^2_\infty = 2\hat{M}^2$ with $\hat{m}^2_D$ and $\hat{M}^2$ given by (5.18) and (5.28) respectively. While the HTL self-energies are no longer accurate at large momenta, their light-cone limit is exact to one-loop order and thus still determines the leading-order asymptotic masses [317, 318].

There are however nevertheless higher-order corrections which in fact require HTL resummation as indicated in (6.17) and (6.18) and which have complicated momentum dependence that has not been evaluated so far. Presently, only the particular averages (6.19) and (6.20) are known from their relation to the plasmon effect in Φ-derivable thermodynamics as discussed in Sect. 6.3.2.

7.8.2. Longitudinal Plasmons For momenta $\mathbf{k}^2 \gg \omega^2_{\mathrm{pl.}}$, the longitudinal plasmon branch approaches the light-cone, as can be seen in figure 9. From $k^2 = \hat{\Pi}_B(k)$ and (5.16) one finds

$$\omega^2_B(|\mathbf{k}|) \to \mathbf{k}^2 \left(1 + 4\mathrm{e}^{-6\mathbf{k}^2/(e^2T^2)-2}\right) \tag{7.31}$$

with $e^2 = g^2(N + N_f/2)$ in QCD, so the light-cone is approached exponentially as $|\mathbf{k}|$ is increased. If one also calculates the residue, one finds that this goes to zero at the same time, and exponentially so, too. A similar behaviour occurs in the "plasmino" branch of the fermion propagator at momenta $\mathbf{k}^2 \gg \hat{M}^2$.

Instead of QCD, we shall consider the analytically tractable case of massless scalar electrodynamics as a simple toy model with at least some similarities to the vastly more complicated QCD case in that in both theories there are bosonic self-interactions. There are however no HTL vertices in scalar electrodynamics, which makes it possible to carry out a complete momentum-dependent NLO calculation [375].

Comparing HTL values of and NLO corrections to $\Pi_{00} = -\mathbf{k}^2\Pi_B/k^2$, one finds that as $k^2 \to 0$ there are collinear singularities in both:

$$\hat{\Pi}_B(k)/k^2 \to \frac{\hat{m}_D^2}{2\mathbf{k}^2} \ln \frac{\mathbf{k}^2}{k^2} \tag{7.32}$$

diverges logarithmically♯, whereas

$$\delta\Pi_B/k^2 \to -e\frac{\hat{m}^2}{|\mathbf{k}|\sqrt{k^2}} \tag{7.33}$$

(with $\hat{m} \propto eT$ the thermal mass of the scalar). Because (7.33) diverges stronger than logarithmically, one has $\delta\Pi_B > \hat{\Pi}_B$ eventually as $k^2 \to 0$. Clearly, this leads to a breakdown of perturbation theory in the immediate neighbourhood of the light-cone ($k^2/|\mathbf{k}|^2 \lesssim (e/\ln\frac{1}{e})^2$), which this time is not caused by the massless magnetostatic modes, but rather by the massless hard modes contained in the HTL's.

However, a self-consistent gap equation for the scalar thermal mass implies that also the hard scalar modes have a thermal mass $\sim eT$. Including this by extending the resummation of the scalar thermal mass to hard internal lines renders Π_B/k^2 finite in the light-cone limit, with the result

$$\begin{aligned}\lim_{k^2\to 0}\frac{\Pi_B^{\text{resum.}}}{k^2} &= \frac{e^2T^2}{3\mathbf{k}^2}\Big[\ln\frac{2T}{\hat{m}} + \frac{1}{2} - \gamma_E + \frac{\zeta'(2)}{\zeta(2)} + O(e)\Big] \\ &= \frac{e^2T^2}{3\mathbf{k}^2}\Big[\ln\frac{2.094\ldots}{e} + O(e)\Big]\end{aligned} \tag{7.34}$$

such that there is a solution to the dispersion law with $k^2 = 0$ at $\mathbf{k}^2/(e^2T^2) = \frac{1}{3}\ln(2.094/e) + O(e)$. Because all collinear singularities have disappeared, continuity implies that there are also solutions for space-like momenta $k^2 < 0$, so the longitudinal plasmon branch pierces the light-cone, having group velocity $v < 1$ throughout, though, as shown in figure 14. While at HTL level, the strong Landau damping at $k^2 < 0$ switches on discontinuously, it now does so smoothly through an extra factor $\exp[-e\sqrt{|\mathbf{k}|/[8(|\mathbf{k}| - \omega)]}]$, removing the longitudinal plasmons through over-damping for $(|\mathbf{k}| - \omega)/|\mathbf{k}| \gtrsim e^2$. This is in fact an essential singularity in the self-energy now, which, as has been shown in reference [403] occurs generally at $k_0 = |\mathbf{k}|$ when the hard degrees of freedom have nonzero mass.

So the collinear singularities that spoil HTL-resummed perturbation theory on the light-cone are associated with a slight but nevertheless qualitative change of the spectrum of longitudinal plasmons: instead of being time-like throughout and existing for higher momenta, albeit with exponentially small and decreasing residue and effective mass, they become space-like at a particular point $|\mathbf{k}|_{\text{crit.}} \sim eT\ln\frac{1}{e}$. For

♯ This is in fact the technical reason why the longitudinal branch approaches the light-cone exponentially when $\mathbf{k}^2 \gg \omega_{\text{pl.}}^2$.

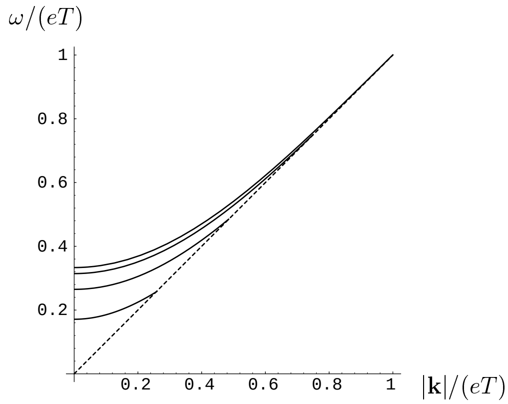


Figure 14. The longitudinal plasmon branch of scalar electrodynamics including NLO corrections to the HTL result. The upper of the four lines gives the HTL result and the lines below correspond to NLO corrections with $e = 0.3$, 1, and 2, respectively. The latter three lines cross the light-cone such that the phase velocity starts to exceed 1, but with group velocity < 1 throughout. In the space-like region, the plasmon modes are damped by Landau damping, which is strong except in the immediate neighbourhood of the light-cone, where it is suppressed by a factor of $\exp\{-e\sqrt{\mathbf{k}/[8(\mathbf{k}-\omega)]}\}$

$|\mathbf{k}| > |\mathbf{k}|_{\mathrm{crit.}}$ Landau damping sets in smoothly but rapidly, so that these modes soon become over-damped as $|\mathbf{k}|$ is increased.

This phenomenon is in fact known to occur in non-ultrarelativistic ($T < m_e$) QED [404], and has been considered in the case of QCD by Silin and Ursov [405], who speculated that it may lead to Cherenkov-like phenomena in the quark-gluon plasma.

In QCD, the situation is in fact much more complicated. Under the assumption that the collinear singularities are removed solely by the resummation of asymptotic gluonic and fermionic thermal masses in hard internal lines, the value of $|\mathbf{k}|$ where longitudinal plasmons turn space-like has been calculated in [375]. For a pure-glue plasma, it reads

$$\mathbf{k}^2_{\mathrm{crit.}} = g^2 T^2 [\ln \frac{1.48\ldots}{g} + \ldots]. \tag{7.35}$$

Such an extended resummation can in fact be related to an improved and still gauge-invariant version of the HTL effective action [318]. However it is likely that damping effects are of equal importance here (in contrast to scalar electrodynamics), so that in (7.35) only the coefficient of the logarithm is complete, but not the constant under the log.

8. Resummations beyond hard thermal loops

In the previous section we have considered perturbative corrections to dynamical quantities at soft scales $\sim gT$. As soon as there is a sensitivity to "ultrasoft" scales $\sim g^2 T$, (HTL) perturbation theory breaks down and typically only leading logarithms can be computed as we have discussed. This is in particular the case when external momenta are either ultrasoft or very close to the light-cone.

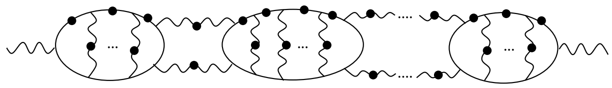


Figure 15. Generalized ladder diagrams contributing to ultrasoft amplitudes.

In the following we shall briefly discuss a few such cases where resummations of the perturbative series have to include more than the HTL diagrams and where important progress has been achieved recently.

8.1. Ultrasoft amplitudes

For ultrasoft external momenta, the infrared sensitivity may in fact become so large that HTL diagrams are no longer of leading order. The diagrams of figure 12 have in fact been evaluated in [406] for $k_0, |\mathbf{k}| \lesssim g^2T$ with the result

$$\delta\Pi_{\mu\nu}(k) = -i\hat{m}_D^2 N g^2 T k_0 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} \frac{v_\mu v'_\nu}{(v\cdot k)(v'\cdot k)} \left[I(\mathbf{v},\mathbf{v}') \ln \frac{gT}{\lambda} + \text{finite} \right] \quad (8.1)$$

with $\lambda \ll gT$ and

$$I(\mathbf{v},\mathbf{v}') = -\delta^{(S_2)}(\mathbf{v}-\mathbf{v}') + \frac{1}{\pi^2} \frac{(\mathbf{v}\cdot\mathbf{v}')^2}{\sqrt{1-(\mathbf{v}\cdot\mathbf{v}')^2}}. \quad (8.2)$$

For $k_0, |\mathbf{k}| \sim g^2T$ this is of the same order than the HTL self-energy $\hat{\Pi}_{\mu\nu} \sim \hat{m}_D^2$ and even logarithmically enhanced.

This strong infrared sensitivity is entirely due to the HTL parts of the dressed vertices in figure 12, with contributions involving tree-level vertices being suppressed by powers of g. By blowing up the HTL vertices in figure 12, one may view the diagram involving one 4-vertex as one hard loop with one soft propagator insertion so that the largeness of (8.1) indeed means that corrections within a HTL diagram cease to be perturbative. This is in fact special to nonabelian gauge fields and absent in Abelian theories (where there are no HTL vertices involving exclusively gauge fields to build the diagrams of figure 12); in Abelian theories there are cancellations between self-energy and vertex corrections [407, 375, 408, 409], which however do not carry over to the nonabelian case.

Moreover, it turns out that there are infinitely many diagrams that need to be taken into account in order to obtain the leading order terms in ultrasoft amplitudes. In the case of the gluon self-energy, such diagrams are generalized ladder diagrams as shown in figure 15.

In order to characterize the leading contributions to ultrasoft amplitudes, a diagrammatic approach is no longer practicable, but it turns out that one can generalize the kinetic theory treatment of HTL's to ultrasoft amplitudes. This requires the inclusion of a collision term which is proportional to the colour relaxation time $\tau_{col} \sim 1/\gamma$, where γ is given by (7.19) in the large-momentum limit ($v_A = 1$).

In leading-logarithmic approximation the required nonabelian Boltzmann equation which generalizes the nonabelian Vlasov equation (6.8) has the form [410, 411, 412, 413, 414, 415]

$$[v\cdot D, W(x,\mathbf{v})]^a = \mathbf{v}\cdot\mathbf{E}^a(x) + \gamma \int d\Omega_{\mathbf{v}'} I(\mathbf{v},\mathbf{v}') W(x,\mathbf{v}'), \quad (8.3)$$

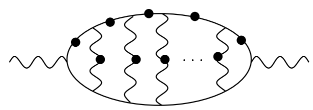


Figure 16. Ladder diagrams contributing to photon production at leading order. In a quark-gluon plasma, the external lines correspond to on-shell photons, the internal lines to dressed quark and gluon propagators. The quarks are hard but collinear with the external photon and exchange soft gluons such as to maintain collinearity.

with $I(\mathbf{v},\mathbf{v}')$ defined in (8.2). Beyond the leading-log approximation [415, 416, 417, 394] the quantity $\gamma I(\mathbf{v},\mathbf{v}')$ needs to be replaced by a collision operator that involves also the HTL gauge-boson self-energies. The ultrasoft vertex functions obtained formally in analogy to the expansion (6.9) however still share many of the remarkable properties of the hard thermal loops, namely gauge-fixing independence and simple Ward identities [415].

At still smaller scales, $\omega \ll |\mathbf{k}| \ll \gamma$, it is in fact possible to simplify the above effective theory of ultrasoft modes and to construct a local effective theory (Bödeker's effective theory) [410, 411, 418], because on distances $|\mathbf{k}|^{-1}$ much larger than the colour coherence length γ^{-1} there are no longer propagating modes with definite colour. Bödeker's effective theory is a stochastic theory given by the Langevin equation

$$\sigma_c \mathbf{E}^a = (\mathbf{D}\times\mathbf{B})^a + \boldsymbol{\zeta}^a, \quad \langle \zeta^{ia}(x)\zeta^{jb}(x')\rangle = 2\sigma_c T \delta^{ij}\delta^{ab}\delta^4(x-x'), \tag{8.4}$$

where $\boldsymbol{\zeta}$ is a Gaussian noise term and σ_c the colour conductivity given to leading order by (7.22); beyond leading-log order, this parameter has been determined in reference [394]. The effective theory (8.4) is also UV finite and was put to use in the numerical calculation [419] of the rate of baryon number violation [122] in the hot symmetry-restored phase of electroweak theory, which at leading order is governed by nonperturbative nonabelian gauge field fluctuations with spatial momenta $\sim g^2T$ and frequencies $\sim g^4T\ln(1/g)$.

8.2. Light-like external momenta

As we have seen in the example of NLO corrections to the dispersion law of longitudinal plasmons in section 7.8.2, the HTL perturbation theory becomes insufficient not only for ultrasoft momenta $|\mathbf{k}| \lesssim g^2T$, but also when amplitudes involve harder momenta that are however nearly light-like, which gives rise to collinear singularities. This problem has surfaced in particular in the calculation of the production rate of real (non-thermalized) photons in a quark-gluon plasma from HTL-resummed perturbation theory [420, 421, 422] and it turned out that damping effects on hard internal lines have to be included as the dominant regulator of collinear singularities [423].

Moreover, for a complete leading-order calculation it turns out to be necessary to resum all ladder diagrams built from dressed propagators of the form shown in figure 16. This reflects the necessity to take into account the physical phenomenon of Landau-Pomeranchuk-Migdal suppression†† of photon emission [425]. However, contrary to the conclusion arrived at in reference [426], thermal photon production from a QCD plasma was shown in reference [427] not to be sensitive to ultrasoft nonperturbative physics, and a complete leading-order evaluation of the photon

†† For a review in the context of energy loss calculations see e.g. reference [424]

production rate from a hot quark-gluon plasma was finally accomplished in references [428, 429]. (The analogous problem in dilepton production was recently solved in reference [430].)

8.3. Transport coefficients

Another important example where an expansion in number of loops breaks down is in transport coefficients, which in linear response theory are given by Kubo relations [431] through correlation functions in the limit of vanishing frequency and at zero spatial momentum of the form

$$\lim_{\omega\to 0}\frac{1}{\omega}\int d^4x e^{i\omega t}\left\langle[\mathcal{O}(t,\mathbf{x}),\mathcal{O}(0)]\right\rangle , \tag{8.5}$$

where $\mathcal{O}$ represents components of a conserved current or of the stress tensor.

In scalar field theory, the required resummation (at leading order) has been worked out in references [432, 433, 434]. The relevant diagrams turn out to be uncrossed ladder diagrams, where each rung brings a pair of propagators with pinching singularities cut off by their thermal width.

In gauge theory, transport coefficients turn out to depend on physics at soft scales such as to require also hard-thermal-loop resummation. For example, the leading term in the shear viscosity has the parametric form $\eta \propto T^3/(g^4\ln(1/g))$, where the logarithm is due to the dynamical screening provided by the HTL gauge propagator [435, 436], and an analogous leading logarithm appears in the various diffusion constants [437]. The initial calculations of these quantities in kinetic theory however turned out to be incomplete already to leading-logarithmic accuracy and have been completed only recently in [438].

In reference [439] a purely diagrammatic rederivation of the relevant integral equation starting from the Kubo formula has been given, which is however incomplete as far as Ward identities are concerned; its completion in the Abelian case has been discussed in [440, 441]. To organize a diagrammatic calculation of transport coefficients, the 2PI approximation scheme mentioned in section 3.5 is an efficient means. It turns out that the 3-loop Φ-derivable approximation is necessary and sufficient to obtain the leading-order results for shear viscosity in a scalar theory [442, 443], and to leading-log order in QED [443].

Whereas at leading-log accuracy the diagrams that need to be resummed are dressed uncrossed ladder diagrams like those of figure 16, the set of diagrams that contribute in a complete leading-order calculation is still much more complicated. Their resummation has nevertheless been achieved recently by means of a Boltzmann equation with a collision term evaluated to sufficient accuracy and with the resulting integral equation treated by variational techniques. The results obtained within this effective kinetic theory [444] so far comprise shear viscosity, electrical conductivity, and flavour diffusion constants [445] (bulk viscosity, however, turns out to be more difficult and has not yet been determined even to leading-log accuracy).

Nonperturbative (all order in g) results for transport coefficients in a gauge theory have been derived for the toy model of large-flavour-number QED or QCD in reference [68] allowing for a test of the quality of HTL approximations, which turn out to work remarkably well up to the point where the renormalization-scale dependence becomes the dominant uncertainty.

9. Hard thermal loops and gravity

In ultrarelativistic field theory, the HTL approximation to the polarization tensor of gauge bosons determines the leading-order spectrum of gauge-field quasi-particles and the linear response of an ultrarelativistic plasma to external perturbations.

In the physics of the very early universe, which is filled with a hot plasma of various elementary particles, the gravitational polarization tensor is also of interest. It describes the (linear) response to metric perturbations and its infrared behaviour determines the evolution of large-scale cosmological perturbations, which provide the seeds for structure formation that are nowadays being studied directly through the anisotropies of the cosmic microwave background and which are being measured with stunning accuracy [446, 447]. The gravitational polarization tensor is also a central quantity in the theory of stochastic (semiclassical) gravity [448, 449], which aims at a general self-consistent description of quantum statistical fluctuations of matter in a curved background geometry as a stepping-stone towards a full quantum theory of gravity.

9.1. HTL gravitational polarization tensor

If Γ denotes all contributions to the effective action besides the classical Einstein-Hilbert action, the energy-momentum tensor is given by the one-point 1PI vertex function

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-g(x)}}\frac{\delta\Gamma}{\delta g^{\mu\nu}(x)} \tag{9.1}$$

and the gravitational polarization tensor by the two-point function

$$-\Pi_{\mu\nu\alpha\beta}(x,y) \equiv \frac{\delta^2\Gamma}{\delta g^{\mu\nu}(x)\delta g^{\alpha\beta}(y)} = \frac{1}{2}\frac{\delta\left(\sqrt{-g(x)}T_{\mu\nu}(x)\right)}{\delta g^{\alpha\beta}(y)}. \tag{9.2}$$

From the last equality it is clear that $\Pi^{\alpha\beta\mu\nu}$ describes the response of the (thermal) matter energy-momentum tensor to perturbations in the metric. Equating $\Pi^{\alpha\beta\mu\nu}$ to the perturbation of the Einstein tensor gives self-consistent equations for metric perturbations and, in particular, cosmological perturbations.

In the ultrarelativistic limit where all bare masses can be neglected and in the limit of temperature much larger than spatial and temporal variations, the effective (thermal) action is conformally invariant, that is $\Gamma[g] = \Gamma[\Omega^2 g]$ (the conformal anomaly like other renormalization issues can be neglected in the high-temperature domain).

This conformal invariance is crucial for the application to cosmological perturbations for two reasons. Firstly, it allows us to have matter in thermal equilibrium despite a space-time dependent metric. As long as the latter is conformally flat, $ds^2 = \sigma(\tau, \mathbf{x})[d\tau^2 - d\mathbf{x}^2]$, the local temperature on the curved background is determined by the scale factor σ. Secondly, the thermal correlation functions are simply given by the conformal transforms of their counterparts on a flat background, so that ordinary momentum-space techniques can be employed for their evaluation.

The gravitational polarization tensor in the HTL approximation has been first calculated fully in [450], after earlier work [451, 452, 453, 454] had attempted (in vain) to identify the Jeans mass (a negative Debye mass squared signalling gravitational instability rather than screening of static sources) from the static limit of the momentum-space quantity $\tilde{\Pi}_{\mu\nu\alpha\beta}(k)$, $k_0 = 0$, $\mathbf{k} \to 0$ in flat space.

Like the HTL self-energies of QED or QCD, $\Pi_{\mu\nu\alpha\beta}$ is an inherently nonlocal object. Because it is a tensor of rank 4, and the local plasma rest frame singles out the time direction, it has a much more complicated structure. From $\eta^{\mu\nu}$, $u_\mu = \delta^0_\mu$ and k_μ one can build 14 tensors to form a basis for $\tilde{\Pi}_{\mu\nu\alpha\beta}(k)$. Its HTL limit $(k_0, |\mathbf{k}| \ll T)$, however, satisfies the Ward identity

$$4k^\mu \tilde{\Pi}_{\mu\nu\alpha\beta}(k) = k_\nu T_{\alpha\beta} - k^\sigma \left(T_{\alpha\sigma}\eta_{\beta\nu} + T_{\beta\sigma}\eta_{\alpha\nu}\right) \tag{9.3}$$

corresponding to diffeomorphism invariance as well as a further one corresponding to conformal invariance, the "Weyl identity"

$$\eta^{\mu\nu}\tilde{\Pi}_{\mu\nu\alpha\beta}(k) = -\frac{1}{2}T_{\alpha\beta}, \tag{9.4}$$

where $T_{\alpha\beta} = P_0(4\,\delta^0_\mu\delta^0_\nu - \eta_{\mu\nu})$ and P_0 the ideal-gas pressure $\propto T^4$. These identities reduce the number of independent structure functions to three, which may be chosen as

$$\Pi_1(k) \equiv \tilde{\Pi}_{0000}(k)/\rho, \quad \Pi_2(k) \equiv \tilde{\Pi}_{0\mu}{}^{\mu}{}_0(k)/\rho, \quad \Pi_3(k) \equiv \tilde{\Pi}_{\mu\nu}{}^{\mu\nu}(k)/\rho, \tag{9.5}$$

where $\rho = T_{00} = 3P_{SB}$.

The HTL limit is in fact universal: the thermal matter may be composed of any form of ultrarelativistic matter such as also gravitons, which are equally important as any other thermalized matter if the graviton background has comparable temperature. It reads[450]

$$\hat{\Pi}_1(k) = \frac{k^0}{2|\mathbf{k}|}\ln\frac{k^0+|\mathbf{k}|}{k^0-|\mathbf{k}|} - \frac{5}{4}, \quad \hat{\Pi}_2 = -1, \quad \hat{\Pi}_3 = 0. \tag{9.6}$$

As in ordinary hot gauge theories, the entire HTL effective action is determined by the Ward identities and has been constructed in [455, 456]. The 3-graviton HTL vertex has moreover been worked out explicitly in [457, 458] and subleading corrections beyond the HTL approximation have been considered in [459].

From (9.6) one can formally derive an HTL-dressed graviton propagator which contains three independent transverse-traceless tensors. As the HTL propagators in gauge theories, this describes collective phenomena in the form of nontrivial quasi-particle dispersion laws and Landau damping (from the imaginary part of $\hat{A}$). In the context of cosmological perturbations, the latter corresponds to the collisionless damping studied in [460], as we shall further discuss below.

In the limit $k_0^2, \mathbf{k}^2 \gg GT^4$, where G is the gravitational coupling constant, one can ignore any background curvature terms $G\rho \propto GT^4$, and use the momentum-space propagator to read off the dispersion relations for the three branches of gravitational quasiparticles. One of these, the spatially transverse-traceless branch, corresponds to the gravitons of the vacuum theory. Denoting this branch by $\mathcal{A}$, (9.6) implies an asymptotic thermal mass for the gravitons according to [450]

$$\omega_{\mathcal{A}}^2 \to k^2 + m_{\mathcal{A}\infty}^2 = k^2 + \frac{5}{9} \times 16\pi G\rho. \tag{9.7}$$

While in a linear response theory branch $\mathcal{A}$ neither couples to perturbations in the energy density δT^{00} nor the energy flux δT^{0i}, the two additional branches $\mathcal{B}$ and $\mathcal{C}$ both couple to energy flux, but only $\mathcal{C}$ to energy density. The additional branches correspond to excitations that are purely collective phenomena, and like the longitudinal plasmon in gauge theories, they disappear from the spectrum as $k^2/(GT^4) \to \infty$. In contrast to ordinary gauge theories, however, they do so by

acquiring effective thermal masses that grow without bound, while at the same time the residues of the corresponding poles disappear in a power-law behaviour (instead of becoming massless and disappearing exponentially) [450]. (Some subleading corrections to these asymptotic masses at one-loop order have been determined in [461], but these are gauge dependent and therefore evidently incomplete.)

For long wavelengths $\mathbf{k}^2 \sim GT^4$ the momentum-space HTL graviton propagator exhibits an instability, most prominently in mode $\mathcal{C}$, which is reminiscent of the gravitational Jeans instability and which is only to be expected since gravitational sources unlike charges cannot be screened. However, this instability occurs at momentum scales which are comparable with the necessarily nonvanishing curvature $R \sim GT^4$. The flat-space graviton propagator is therefore no longer the appropriate quantity to analyse; metric perturbations have to be studied in a curved time-dependent background.

9.2. Selfconsistent cosmological perturbations from thermal field theory

The conformal covariance of the HTL gravitational polarization tensor $\hat\Pi_{\mu\nu\alpha\beta}(x,y)$ as expressed by the Weyl identity (9.4) determines this non-local quantity in a curved but conformally flat space according to

$$\hat\Pi_{\mu\nu\alpha\beta}(x,y)\Big|_{g=\sigma\eta} = \sigma(x)\sigma(y)\hat\Pi_{\mu\nu\alpha\beta}(x-y)\Big|_{g=\eta}. \tag{9.8}$$

A closed set of equations for metric perturbations is obtained from using this in the right-hand side of the Einstein equation linearized around a conformally-flat background cosmological model [450, 462, 463]

$$\begin{aligned}\delta G^{\mu\nu} &\equiv \frac{\delta(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R)}{\delta g^{\alpha\beta}}\delta g^{\alpha\beta} = -8\pi G\,\delta T^{\mu\nu} = -8\pi G\int_{x'}\frac{\delta T^{\mu\nu}(x)}{\delta g_{\alpha\beta}(x')}\delta g_{\alpha\beta}(x') \\ &= 4\pi G\left[T^{\mu\nu}g^{\alpha\beta}+2T^{\mu\alpha}g^{\beta\nu}\right]\delta g_{\alpha\beta}(x) + \frac{16\pi G}{\sqrt{-g(x)}}\int_{x'}\Pi^{\mu\nu\alpha\beta}(x,x')\delta g_{\alpha\beta}(x'). \end{aligned} \tag{9.9}$$

In a hydrodynamic approach, $\delta T^{\mu\nu}$ is usually determined by certain equations of state together with covariant conservation, the simplest case of which is that of a perfect fluid, which has been studied in the pioneering work of Lifshitz [464, 465]. Many generalizations have since been worked out and cast into a gauge invariant form by Bardeen [466] (see also [467, 468]).

Relativistic and (nearly) collisionless matter, however, has more complicated gravitational interactions than a perfect fluid. This is usually studied using classical kinetic theory [469, 470] and the case of purely collisonless matter has been worked out in [471] with some numerical solutions obtained in [472, 460, 473] for particular gauge choices.

9.2.1. Purely collisionless matter In [462], the self-consistent equations for (scalar) density perturbations of a radiation-dominated Robertson-Walker-Friedmann model with collisionless matter have been derived using the HTL gravitational polarization tensor. By virtue of the diffeomorphism Ward identity (9.3), these equations turn out to be automatically gauge independent.

For example, in the simple radiation-dominated and spatially-flat Einstein-de Sitter model

$$ds^2 = \sigma(\tau)(d\tau^2 - d\mathbf{x}^2), \qquad \sigma(\tau) = \frac{8\pi G\rho_0}{3}\tau^2, \tag{9.10}$$

where ρ_0 is the energy density when $\sigma = 1$ and τ is the conformal time (which equals the size of the Hubble horizon in comoving coordinates), the scalar part of metric perturbations can be parametrized in terms of four scalar functions

$$\delta g^{(S)}_{\mu\nu} = \sigma(\tau) \begin{pmatrix} C & D_{,i} \\ D_{,j} & A\delta_{ij} + B_{,ij} \end{pmatrix}. \tag{9.11}$$

Of these, two can be gauged away by diffeomorphisms. But in a gauge-invariant framework only gauge-invariant combinations enter nontrivially. Only two independent gauge-invariant combinations exist, which may be chosen as

$$\Phi = A + \frac{\dot\sigma}{\sigma}(D - \tfrac{1}{2}\dot B) \tag{9.12}$$

$$\Pi = \tfrac{1}{2}(\ddot B + \frac{\dot\sigma}{\sigma}\dot B + C - A) - \dot D - \frac{\dot\sigma}{\sigma}D, \tag{9.13}$$

where a dot denotes differentiation with respect to the conformal time variable τ.

Each spatial Fourier mode with wave vector $\mathbf{k}$ is related to perturbations in the energy density and anisotropic pressure according to

$$\delta = \frac{1}{3}x^2\Phi, \qquad \pi_{\text{anis.}} = \frac{1}{3}x^2\Pi, \tag{9.14}$$

where

$$x \equiv k\tau = \frac{R_H}{\lambda/(2\pi)}, \tag{9.15}$$

which measures the (growing) size of the Hubble horizon over the wavelength of a given mode (which is constant in comoving coordinates). In (9.14) energy density perturbations δ are defined with respect to space-like hypersurfaces representing everywhere the local rest frame of the full energy-momentum tensor, whereas $\pi_{\text{anis.}}$ is an unambiguous quantity, since there is no anisotropic pressure in the background.

Correspondingly, when specifying to scalar perturbations, there are just two independent equations contained in (9.9). Because of conformal invariance, the trace of (9.9) is particularly simple and yields a finite-order differential equation in x,

$$\Phi'' + \frac{4}{x}\Phi' + \frac{1}{3}\Phi = \frac{2}{3}\Pi - \frac{2}{x}\Pi' \tag{9.16}$$

(a prime denotes differentiation with respect to the dimensionless time variable x). The other components, however, involve the nonlocalities of the gravitational polarization tensor. These lead to an integro-differential equation, which upon imposing retarded boundary conditions reads [462]

$$(x^2 - 3)\Phi + 3x\Phi' = 6\Pi - 12\int_{x_0}^{x} dx'\, j_0(x - x')[\Phi'(x') + \Pi'(x')] + \varphi(x - x_0) \tag{9.17}$$

where $j_0(x) = \sin(x)/x$ arises as Fourier transform of $\hat\Pi_1(\omega/k)$ in (9.6). $\varphi(x - x_0)$ encodes the initial conditions, the simplest choice of which corresponds to $\varphi(x - x_0) \propto j_0(x - x_0)$.

Similar integro-differential equations have been obtained from coupled Einstein-Vlasov equations in particular gauges, and the above one can be shown to arise from a gauge-invariant reformulation of classical kinetic theory [474]. Usually, such equations are studied numerically, with only some asymptotic behaviour having been analysed analytically [475, 476]. Remarkably enough, they can be solved analytically [462] provided initial conditions are formulated for $x_0 \to 0$. In this case a power series

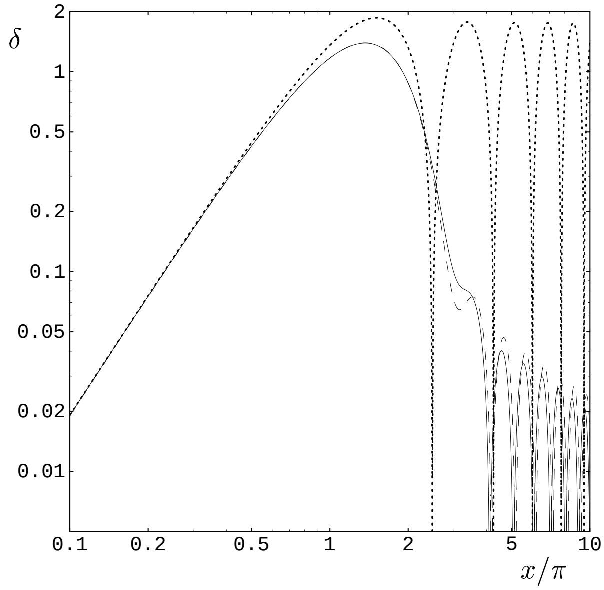


Figure 17. The energy-density contrast (arbitrary normalization) as a function of x/π for a collisionless ultrarelativistic plasma (full line), a scalar plasma with quartic self-interactions $\lambda\phi^4$ and $\lambda = 1$ (dashed line), and a perfect radiation fluid (dotted line).

ansatz for Φ and Π leads to solvable recursion relations for an alternating series that converges faster than trigonometric functions.

This also holds true for the vector (rotational) and tensor perturbations and when the more realistic case of a two-component system of a perfect radiation fluid combined with a collisionless ultrarelativistic plasma is considered [463]. In the case of rotational perturbations these studies led to novel solutions not considered before [477].

In figure 17, the solution for the energy-density contrast is given in a doubly-logarithmic plot (full line) and compared with the perfect-fluid case (dotted line). In the latter, one has growth of the energy-density contrast as long as the wavelength of the perturbation exceeds the size of the Hubble radius ($x \ll 1$). After the Hubble horizon has grown such as to encompass about one half wavelength ($x = \pi$), further growth of the perturbation is stopped by the strong radiation pressure, turning it into an (undamped) acoustic wave propagating with the speed of sound in radiation, $v = 1/\sqrt{3}$. The collisionless case is similar as concerns the superhorizon-sized perturbations, but after horizon crossing, there is strong damping $\sim 1/x$, and the phase velocity is about 1. This indeed reproduces the findings of the numerical studies of reference [460]. They can be understood as follows: a energy-density perturbation consisting of collisionless particles propagates with the speed of their constituents, which in the ultrarelativistic case is the speed of light, and there is collisionless damping in the form of directional dispersion.

Tensor perturbations, which correspond to primordial gravitational waves, are also modified by a nearly collisionless background component as worked out in references [463, 474].

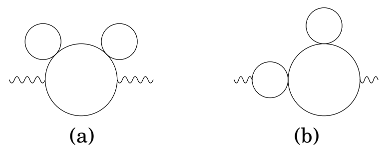


Figure 18. Two examples of infrared divergent contributions to the gravitational polarization tensor in scalar ϕ^4 theory beyond two-loop order.

9.2.2. Weak self-interactions and HTL resummation The thermal-field-theory treatment of the effects of an ultrarelativistic plasma on the evolution of cosmological perturbations can be used also to study the effect of weak self-interactions in the plasma by calculating higher-order contributions to the gravitational polarization tensor. A virtue of this approach is that everything is formulated in purely geometrical terms, without explicit recourse to perturbations in the (gauge variant) distribution functions of a kinetic-theory treatment.

In reference [63], the gravitational polarization tensor has been calculated in a $\lambda\phi^4$ theory through order $\lambda^{3/2}$. The next-to-leading order contributions to $\Pi_{\mu\nu\alpha\beta}$ at order λ^1 are contained in the high-temperature limit of two-loop diagrams and their evaluation is straightforward. However, starting at three-loop order, there are infrared divergences which signal a breakdown of the conventional perturbative series. This is caused by the generation of a thermal mass $\propto \sqrt{\lambda}T$ for the hot scalars. If this is not resummed into a correspondingly massive scalar propagator, repeated insertions of scalar self-energy diagrams in a scalar line produces arbitrarily high powers of massless scalar propagators all with the same momentum, and thus increasingly singular infrared behaviour (figure 18a).

However, it is not sufficient to resum this thermal mass for the hot scalars, as this would break conformal invariance (which in the ultrarelativistic case is only broken by the nonthermal conformal anomaly). Indeed, there are also vertex subdiagrams $\propto \lambda T^2$ that have a similar effect as a self-energy insertion, see figure 18b. As in the hard-thermal-loop resummation program for ordinary gauge theories, one has to resum also nonlocal vertex contributions. Doing so, the result turns out to satisfy both the diffeomorphism and conformal Ward identities.

In the low-momentum limit of interest in the theory of cosmological perturbations, the function Π_1 in (9.6) that governs the evolution of scalar perturbations reads, to order $\lambda^{3/2}$

$$\begin{aligned}\Pi_1(\varpi) &= \varpi\,\mathrm{artanh}\frac{1}{\varpi} - \frac{5}{4} + \frac{5\lambda}{8\pi^2}\left[2\left(\varpi\,\mathrm{artanh}\frac{1}{\varpi}\right)^2 - \varpi\,\mathrm{artanh}\frac{1}{\varpi} - \frac{\varpi^2}{\varpi^2-1}\right] \\ &\quad + \frac{5\lambda^{3/2}}{8\pi^3}\left[3\left(\varpi^2 - 1 - \varpi\sqrt{\varpi^2-1}\right)\left(\varpi\,\mathrm{artanh}\frac{1}{\varpi}\right)^2\right. \\ &\qquad + 6\left(\varpi\sqrt{\varpi^2-1} - \varpi^2 - \frac{\varpi}{\sqrt{\varpi^2-1}}\right)\varpi\,\mathrm{artanh}\frac{1}{\varpi} \\ &\qquad \left. + \frac{\varpi}{(\varpi^2-1)^{3/2}} + 3\frac{\varpi^2}{\varpi^2-1} + 6\frac{\varpi}{\sqrt{\varpi^2-1}} - 3\varpi\sqrt{\varpi^2-1} + 3\varpi^2\right]\end{aligned} \tag{9.18}$$

where $\varpi \equiv k_0/|\mathbf{k}|$, and similarly complicated expressions arise for Π_2 and Π_3, which in the collisionless limit (9.6) are simple constants.

The Fourier transform of this expression determines the kernel in the convolution integral of (9.17). At order λ^1, it can still be expressed in terms of well-known special

functions [478], whereas at order $\lambda^{3/2}$ this would involve rather intractable integrals over Lommel functions. However, all that is needed for finding analytical solutions is their power series representations which are comparatively simple. Given them, it is as easy as before to solve the perturbation equations. However, one finds that the asymptotic behaviour $x \gg 1$ eventually becomes sensitive to higher and higher loop orders. The reason for this is that higher loop orders come with increasingly singular contributions at $\varpi = \pm 1$ to $\Pi(\varpi)$, and the large-x behaviour is dominated by the latter. This could be cured by a further resummation similar to the one considered in the case of the dispersion relations of longitudinal plasmons in the vicinity of the light-cone in (7.34), but it turns out that a particular Padé-approximant based on the perturbative result reflects the effects of this further resummation quite well [479]. The result for the density perturbations in a scalar plasma with $\lambda\phi^4$-interactions and $\lambda = 1$ are shown in figure 17 by the dashed line, where it is compared with the collisionless case (full line) and the one of a perfect radiation fluid (dotted line). The effects of the self-interactions within the ultrarelativistic plasma become important only for $x \gtrsim \pi$, where the strong collisionless damping is somewhat reduced and the phase velocity is smaller than 1.

10. Conclusion

In this review we have concentrated on the progress made during the last decade in calculating by analytic means static equilibrium and dynamic near-equilibrium properties in ultrarelativistic gauge theories. Even when the practical utility of the obtained results is often open to debate, they are hopefully paving the way for understanding the more complicated realistic situations to be analysed in present or future heavy-ion experiments, or in astrophysical problems.

Static quantities at high temperature and not too high chemical potential can be efficiently investigated by lattice simulations, whereas perturbative methods seemed to be too poorly convergent to be of any predictive power in the applications of interest. However, in the last few years there have been several different investigations leading to the conclusion that a careful resummation which emphasizes a weakly interacting quasiparticle picture, or, in dimensional reduction, the effective-field-theory aspect, is able to provide dramatic improvements, restoring predictivity even down to a few times the deconfinement temperature in strongly interacting QCD.

In nonabelian gauge theories the perturbative approach is limited by the magnetic mass scale, or, more precisely, the physics of confinement in the chromo-magnetostatic sector. However, weak-coupling effective-field-theory methods can still be used to combine analytical and (numerical) nonperturbative techniques to achieve further progress which is complementary to or perhaps beyond the capacities of a direct numerical approach in four dimensions.

In the case of dynamic properties, and also in the case of static properties when dimensional reduction is not applicable as in cold ultradegenerate plasmas, the analytical approach is even more important, and it is in fact here that the greatest variety of phenomena are encountered. At weak coupling, there are several spatial or temporal scales that need to be distinguished and are at the root of the required resummations in a perturbative treatment. In terms of frequencies and momenta, the first important scale below the hard scale of temperature or chemical potential is the soft scale set by the Debye mass responsible for screening of electric fields as well as for the frequency of long-wavelength plasma oscillations. This is the realm of HTL

resummations, which is however limited by an eventual infrared sensitivity to ultrasoft scales occurring at some (mostly very low) order of the expansion.

Resummations at zero temperature and high chemical potential are not limited by the magnetic mass scale, but nearly static magnetic modes lead to qualitative changes (non-Fermi-liquid behaviour). There is furthermore the nonperturbative phenomenon of colour superconductivity, which is however to some extent accessible by weak-coupling methods, which strongly depend on HDL resummation techniques.

At high temperature, there is, between soft and ultrasoft scales, a further scale set by the damping rate of the hard plasma constituents, which in a nonabelian plasma determines a colour coherence length. The corresponding energy scale is enhanced over the ultrasoft scale by a logarithm in the inverse coupling, which allows for novel systematic developments that we could only cursorily describe.

In the final brief excursion to general relativity, we described the role of the HTL contributions to the gravitational polarization tensor in the theory of cosmological perturbations. There the soft scale is given by the Jeans mass which is in a cosmological situation comparable with the scale of the (inverse) Hubble horizon.

It should be needless to emphasize that many interesting topics in thermal field theory have been covered only cursorily or even not at all. Some of the very recent developments in fact are about to leave the arena of traditional thermal field theory towards a more general, fully nonequilibrium field theory, which is only timely in view of the wealth of experimental of data to be expected from the modern relativistic heavy-ion colliders. But (near-) equilibrium thermal field theory will certainly continue to be an important theoretical laboratory where many physical concepts are brought together for cross-fertilization.

A. Spectral representation of HTL/HDL propagators

A.1. Gauge boson propagator

For the two nontrivial structure functions of the HTL/HDL gauge boson propagator corresponding to the branches A and B it is convenient to separate off the kinematical pole at $k^2=0$ in Δ_B and to define

$$\Delta_t = -\Delta_A = \frac{-1}{k_0^2 - \mathbf{k}^2 - \hat{\Pi}_A(k_0,|\mathbf{k}|)} = \int_{-\infty}^{\infty} \frac{dk_0'}{2\pi} \frac{\hat{\rho}_t(k_0',|\mathbf{k}|)}{k_0'-k_0}, \tag{A.1}$$

$$\Delta_\ell = -\frac{k^2}{\mathbf{k}^2}\Delta_B = \frac{-1}{\mathbf{k}^2 + \hat{\Pi}_B(k_0,|\mathbf{k}|)} = \int_{-\infty}^{\infty} \frac{dk_0'}{2\pi} \frac{\hat{\rho}_\ell(k_0',|\mathbf{k}|)}{k_0'-k_0} - \frac{1}{\mathbf{k}^2}, \tag{A.2}$$

where $\hat{\Pi}_{A,B}$ are the HTL quantities given in (5.15), (5.16). The spectral functions are given by

$$\begin{aligned}\hat{\rho}_{t,\ell}(k_0,|\mathbf{k}|) &= \mathrm{Disc}\,\Delta_{t,\ell}(k_0,|\mathbf{k}|) = 2\lim_{\epsilon\to 0}\,\mathrm{Im}\,\Delta_{t,\ell}(k_0+\mathrm{i}\epsilon,|\mathbf{k}|)\\ &= 2\pi\varepsilon(k_0)\, z_{t,\ell}(|\mathbf{k}|)\,\delta(k_0^2-\omega_{t,\ell}^2(|\mathbf{k}|)) + \beta_{t,\ell}(k_0,|\mathbf{k}|)\,\theta(-k^2)\end{aligned} \tag{A.3}$$

with $\omega_{t,\ell}^2(|\mathbf{k}|)$ as shown in figure 9 (for $\mathbf{k}^2>0$). For small and large values of $\mathbf{k}^2$ they are approximated by [217]

$$\omega_t^2 \simeq \omega_{\rm pl.}^2 + \frac{6}{5}\mathbf{k}^2, \quad \omega_\ell^2 \simeq \omega_{\rm pl.}^2 + \frac{3}{5}\mathbf{k}^2, \qquad \mathbf{k}^2 \ll \omega_{\rm pl.}^2 \tag{A.4}$$

$$\omega_t^2 \simeq \mathbf{k}^2 + m_\infty^2, \quad \omega_\ell^2 \simeq \mathbf{k}^2 + 4\mathbf{k}^2 \exp\left(-\frac{\mathbf{k}^2}{m_\infty^2} - 2\right), \quad \mathbf{k}^2 \gg m_\infty^2 \tag{A.5}$$

where $\omega_{\rm pl.}^2 = \hat{m}_D^2/3$ is the plasma frequency common to both modes, and $m_\infty^2 = \hat{m}_D^2/2$ is the asymptotic mass of transverse quasiparticles. The effective thermal mass of mode ℓ (or B) vanishes exponentially for large $\mathbf{k}^2$.

The residues $z_{t,\ell}$ are defined by

$$z_{t,\ell}^{-1} = \left[\frac{\partial}{\partial k_0^2}(-\Delta_{t,\ell})^{-1}\right]\Big|_{\Delta_{t,\ell}^{-1}=0} \tag{A.6}$$

and explicitly read

$$z_t = \frac{2k_0^2 k^2}{\hat{m}_D^2 k_0^2 - (k^2)^2}\Big|_{k_0=\omega_t(|\mathbf{k}|)}, \quad z_\ell = \frac{2k_0^2 k^2}{\mathbf{k}^2(\hat{m}_D^2 - k^2)}\Big|_{k_0=\omega_\ell(|\mathbf{k}|)} \tag{A.7}$$

with the following asymptotic limits [218]

$$z_t \simeq 1 - \frac{4\mathbf{k}^2}{5\omega_{\rm pl.}^2}, \qquad z_\ell \simeq \frac{\omega_{\rm pl.}^2}{\mathbf{k}^2}\left(1 - \frac{3}{10}\frac{\mathbf{k}^2}{\omega_{\rm pl.}^2}\right), \qquad \mathbf{k}^2 \ll \omega_{\rm pl.}^2 \tag{A.8}$$

$$z_t \simeq 1 - \frac{m_\infty^2}{2\mathbf{k}^2}\left(\ln\frac{4\mathbf{k}^2}{m_\infty^2} - 2\right), \quad z_\ell \simeq \frac{4\mathbf{k}^2}{m_\infty^2}\exp\left(-\frac{\mathbf{k}^2}{m_\infty^2} - 2\right), \quad \mathbf{k}^2 \gg m_\infty^2. \tag{A.9}$$

The singular behaviour of z_ℓ for $\mathbf{k}^2 \to 0$ is in fact only due to the factor $k^2/\mathbf{k}^2$ in (A.2); the residue in Δ_B approaches 1 in this limit. For $\mathbf{k}^2 \gg m_\infty^2$, the residue in Δ_ℓ vanishes exponentially, as mentioned in Section 5.1.2.

The Landau-damping functions $\beta_{t,\ell}$ are given by

$$\beta_t(k_0,|\mathbf{k}|) = \pi\hat{m}_D^2 \frac{k_0(-k^2)}{2|\mathbf{k}|^3}|\Delta_t(k_0,|\mathbf{k}|)|^2, \quad \beta_\ell(k_0,|\mathbf{k}|) = \pi\hat{m}_D^2\frac{k_0}{|\mathbf{k}|}|\Delta_\ell(k_0,|\mathbf{k}|)|^2. \tag{A.10}$$

These are odd functions in k_0 which vanish at $k_0 = 0$ and at $k_0^2 = \mathbf{k}^2$. For large $\mathbf{k}^2$ and fixed ratio $k_0/|\mathbf{k}|$, $\beta_{t,\ell}$ decay like $1/\mathbf{k}^4$.

The spectral functions $\rho_{t,\ell}$ satisfy certain sum rules which can be obtained by a Taylor expansion of (A.1) and (A.2) in k_0 [307, 480, 10]. A special, particularly important case is obtained by putting $k_0 = 0$ in (A.1) and (A.2) which yields

$$\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{\hat\rho_t(k_0,|\mathbf{k}|)}{k_0} = \frac{1}{\mathbf{k}^2}, \quad \int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{\hat\rho_\ell(k_0,|\mathbf{k}|)}{k_0} = \frac{1}{\mathbf{k}^2} - \frac{1}{\mathbf{k}^2+\hat{m}_D^2}. \tag{A.11}$$

More complicated sum rules have been found in applications of HTL resummations. In [481] it has been shown that one can reexpress the integrals involving only the continuous parts of the spectral functions appearing in the energy loss formulae of heavy particles [482] in terms of generalized (Lorentz-transformed) Kramers-Kronig relations; in [483] a sum rule involving also only the Landau damping domain $|x| < 1$ with $x \equiv k_0/|\mathbf{k}|$ has been derived which appears in calculations of the photon or dilepton production rates in a quark-gluon plasma [484]

$$\frac{1}{\pi}\int_0^1 \frac{dx}{x}\frac{2\,{\rm Im}\,\hat\Pi_i(x)}{[t + {\rm Re}\,\hat\Pi_i(x)]^2 + [{\rm Im}\,\hat\Pi_i(x)]^2} = \frac{1}{t+{\rm Re}\,\hat\Pi_i(\infty)} - \frac{1}{t + {\rm Re}\,\hat\Pi_i(0)} \tag{A.12}$$

for $t > 0$ and $i = A, B$.

A peculiar sum rule has been encountered in [99],

$$\int\frac{d^4k}{k_0}\left\{2\,{\rm Im}\,\hat\Pi_A\,{\rm Re}\,\Delta_t - {\rm Im}\,\hat\Pi_B\,{\rm Re}\,\hat\Delta_\ell\right\} = 0 \tag{A.13}$$

which has so far only been shown to hold by numerical integrations. Like (A.12) this sum rule receives contributions only from the Landau damping domain $k^2 < 0$, but it involves the two branches at the same time and it holds only under both k_0 and $|\mathbf{k}|$ integrations.

A.2. Fermion propagator

The spectral representation of the two branches of the HTL fermion propagator is given by

$$\Delta_\pm = \frac{-1}{k_0 \mp (|\mathbf{k}| + \hat{\Sigma}_\pm(k_0, |\mathbf{k}|)} = \int_{-\infty}^{\infty} \frac{dk_0'}{2\pi} \frac{\hat{\rho}_\pm(k_0', |\mathbf{k}|)}{k_0' - k_0} \tag{A.14}$$

where $\rho_\pm$ are defined in analogy to (A.3),

$$\hat{\rho}_\pm(k_0, |\mathbf{k}|) = 2\pi\varepsilon(k_0)\, z_\pm(|\mathbf{k}|)\, \delta(k_0^2 - \omega_\pm^2(|\mathbf{k}|)) + \beta_\pm(k_0, |\mathbf{k}|)\, \theta(-k^2) \tag{A.15}$$

with $\omega_\pm^2(|\mathbf{k}|)$ as shown in figure 10. For small and large values of $\mathbf{k}^2$ they are approximated by [228]

$$\omega_+ \simeq \hat{M} + \frac{|\mathbf{k}|}{3}, \quad \omega_- \simeq \hat{M} - \frac{|\mathbf{k}|}{3}, \qquad \mathbf{k}^2 \ll \hat{M}^2 \tag{A.16}$$

$$\omega_+^2 \simeq \mathbf{k}^2 + M_\infty^2, \quad \omega_- \simeq |\mathbf{k}| \left(1 + 2\exp\left(-\frac{\mathbf{4}k^2}{M_\infty^2} - 1 \right) \right), \quad \mathbf{k}^2 \gg M_\infty^2 \tag{A.17}$$

where $M_\infty^2 = 2\hat{M}^2$ is the asymptotic mass of energetic fermions of the (+)-branch. The effective thermal mass of the additional (−)-branch vanishes exponentially for large $\mathbf{k}^2$.

The residues $z_\pm$ are given by the simple expression

$$z_\pm = \frac{\omega_\pm^2 - \mathbf{k}^2}{2\hat{M}^2} \tag{A.18}$$

with the following asymptotic limits [218]

$$z_+ \simeq \frac{1}{2} + \frac{|\mathbf{k}|}{3\hat{M}}, \qquad z_- \simeq \frac{1}{2} - \frac{|\mathbf{k}|}{3\hat{M}}, \qquad \mathbf{k}^2 \ll \hat{M}^2 \tag{A.19}$$

$$z_+ \simeq 1 - \frac{M_\infty^2}{4\mathbf{k}^2} \left(\ln \frac{4\mathbf{k}^2}{M_\infty^2} - 1 \right), \quad z_- \simeq \frac{4\mathbf{k}^2}{M_\infty^2} \exp\left(-\frac{\mathbf{4}k^2}{M_\infty^2} - 1 \right), \quad \mathbf{k}^2 \gg M_\infty^2. \tag{A.20}$$

For $\mathbf{k}^2 \gg M_\infty^2$, the residue in Δ_- vanishes exponentially.

The Landau-damping functions $\beta_\pm$ are given by

$$\beta_\pm = \pi \hat{M}^2 \frac{|\mathbf{k}| \mp k_0}{\mathbf{k}^2} \left| \Delta_\pm(k_0, |\mathbf{k}|) \right|^2. \tag{A.21}$$

For large $\mathbf{k}^2$ and fixed ratio $k_0/|\mathbf{k}|$, they decay like $1/\mathbf{k}^2$.

In contrast to the gauge boson case, the spectral functions are not odd functions in k_0 but rather obey

$$\rho_+(-k_0, |\mathbf{k}|) = \rho_-(k_0, |\mathbf{k}|). \tag{A.22}$$

Sum rules for these spectral functions have been discussed in detail in [10]. A more complicated one that plays a role in the HTL resummed calculation of the hard photon production rate [485, 486] has been given recently in [481].

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